# ALGEBRA;

ADAPTED TO THE

## COURSE OF INSTRUCTION USUALLY PURSUED

IN THE

## Colleges and Academies

OF THE

## UNITED STATES.

BY

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# PREFACE.

It would be easy to state, by way of preface, the precise reasons which have led the author to add this Algebra to the numberless treatises on the same subject already in existence. If, however, the reader will consent to devote a few leisure moments to an examination of the following points, the writer flatters himself with the belief that these reasons will appear much more forcibly than if stated in the language of an argument

Attention is invited to the accuracy of the Definitions; to the brevity and clearness of the demonstrations; the explanation of *positive* and *negative* quantities; the subject of *factoring;* the appropriateness and careful gradation of equations and other problems; the manner in which the transition from the reduction of equations to the solution of problems is effected; the perpetual recurrence of the mind of the pupil, as he advances, to first principles,—as, for instance, compare the subjects of Greatest Common Divisor, Least Common Multiple, Reduction of Fractions and Equations having *equal roots;* to the circumstance that many of the problems have been doubled by placing figures in parentheses corresponding to each other; the constant requirement of reducing generalizations to

numerical problems previously solved; the treatment of Quadratic Equations, particularly those involving two unknown quantities; the subject of Logarithms; and, finally, to the practical manner in which the Higher Equations are treated.

Many other points might be mentioned; but, in passing over the above, the reader will not fail to discover them.

In the preparation of the work, the author has been occupied some ten or twelve years; and he now feels safe in pledging himself that no material change will be made in any of its discussions during, at least, the same length of time. Every part of it has been repeatedly tested in the class-room.

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# ALGEBRA.

## CHAPTER I.

### DEFINITIONS AND EXERCISES.

**1.** ALGEBRA investigates the relations of quantities by SYMBOLS.

#### SYMBOLS OF QUANTITY.

**2.** The *symbols of quantity* in Algebra are the *letters of the alphabet.*

**3.** The first letters of the alphabet, viz., $a$, $b$, $c$, . . . , usually represent quantities whose numerical values are *known.*

**4.** The last letters, viz., $x$, $y$, $z$, represent quantities whose values are *unknown,* — *i. e.,* unknown before the operations in which they are involved are performed; after these operations *unknown* quantities become *known.*

**5.** An *algebraic quantity* is properly, then, a quantity represented by a *letter* or *letters.*

**6.** An *arithmetical quantity* is one represented by a *figure* or *figures.*

**7.** Algebraic quantities are therefore called *literal quantities,* to distinguish them from *numerical quantities.* Both kinds of quantities are used in Algebra.

### SYMBOLS OF OPERATION.

**8.** The sign $+$, *plus*, indicates that the quantity before which it is placed is to be taken *additively*. Thus, $a + b$, $a$ plus $b$, denotes that the quantity $b$ is to be *added* to the quantity $a$.

**9.** The sign $-$, *minus*, indicates that the quantity before which it is placed is to be taken *subtractively*. Thus, $a - b$, $a$ minus $b$, denotes that the quantity $b$ is to be *subtracted* from the quantity $a$.

When no sign is written, $+$ is understood. Thus, $a$ is the same as $+a$.

**10.** There are three ways in which to indicate Multiplication in Algebra, viz., $x \times y$, $x . y$, and $xy$, all of which indicate that $x$ is to be multiplied by $y$.

**11.** There are also three ways in which to indicate Division in Algebra, viz., $x \div y$, $\frac{x}{y}$, and $x | \underline{y}$, all of which signify that $x$ is to be divided by $y$.

**12.** The sign $(\ )$, *parenthesis*, or $\overline{\qquad}$, *vinculum*, which may also be drawn perpendicularly, is used to connect several algebraic expressions, and denotes that they are to be treated as a single expression. Thus,

$$\left.\begin{array}{r} a \\ +b \\ +c \end{array}\right| x, \quad \text{is the same as } \overline{a+b+c}.x, \text{ or } (a+b+c)x.$$

all of which signify that the sum of $a$, $b$, and $c$ is to be multiplied by $x$. Again, $(4 + 5) \times 6$ is the same as 54, but $4 + 5 \times 6$ is the same as 34.

**13.** Of Exponents.—In the expressions $a\ b\ c$; $x\ y\ z$; $m\ n$; etc., each of the letters composing the expression is called a *literal factor*. If a letter is to occur as a factor several times, instead of writing $a\ a$; $x\ x\ x$; $y\ y\ y\ y$; etc., a figure is placed

at the right hand of the letter and a little above; thus, $a^2$; $x^3$; $y^4$; etc., signify that $x$, $y$, and $z$ have been taken as factors, twice, three times, four times, etc. This figure is called an EXPONENT.

1. An exponent may be integral, fractional, positive, or negative.

2. An *integral positive exponent* of a quantity denotes a POWER of that quantity.

3. A *positive fractional exponent* of a quantity denotes a ROOT of that quantity. Thus, $x^4$ is the same as the fourth *power* of $x$, or $x\ x\ x\ x$. But, $x^{\frac{1}{4}}$ is the same as the fourth *root* of $x$.

4. The fractional exponent may combine both a *power* and a *root*. Thus, $x^{\frac{3}{4}}$ is the same as the *fourth root of* $x$ *cube*.

5. A *negative exponent* of a quantity indicates that the *reciprocal* of the quantity is to be taken with the *sign of the exponent changed*. Thus, $x^{-2}$ is the same as $\frac{1}{x^2}$, and $\frac{1}{x^{-\frac{1}{2}}}$ is the same as $x^{\frac{1}{2}}$. (*Vide* **22**, 1.)

6. A letter may represent any exponent: as $x^m$, read $x$ $m^{th}$ power.

7. Roots are also expressed, as in Arithmetic, by the signs $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc. Thus, $x^{\frac{3}{4}}$ is the same as $\sqrt[4]{x^3}$, and $a^{\frac{x}{y}}$ is the same as $\sqrt[y]{a^x}$.

8. When no exponent is expressed, 1 is understood. Thus, $a$ is the same as $a^1$.

9. Any quantity having 0 for an exponent is the same as 1. Thus, $a^0$ is 1. (*Vide* **68**, ex. 1.)

**14.** Of COËFFICIENTS.—Instead of the expression $a + a$, we may write $2a$; for $a + a + a$, we may write $3a$; for $- x - x$, we may write $- 2x$; for $- x - x - x - x - x$, we may write $- 5x$. In each case the figure standing before the letter shows *how many times* the letter is taken additively or subtractively. This figure is called a COËFFICIENT.

1. A coëfficient may be integral, fractional, positive, or negative. Thus, $5x$, $\frac{1}{5}x$, and $-\frac{2}{3}x$.

2. A coëfficient may be represented by a letter; thus, $bx^2$.

3. When no coëfficient is written, 1 is understood; thus, $a$ is the same as $1a$.

4. The expression $0x$ is the same as 0.

**15.** SYMBOLS OF RELATION. — The sign $=$ indicates that the quantities between which it is placed are *equal;* thus, $x = y$ signifies that $x$ equals $y$.

1. The whole expression of which the sign $=$ is a part, is called an EQUATION.

2. That part of an equation on the *left* of the sign $=$ is called the FIRST MEMBER.

3. That part of an equation on the *right* of the sign $=$ is called the SECOND MEMBER. Thus, $2x + 3y = a - 5b + c$ is an equation of which $2x + 3y$ is the first member, and $a - 5b + c$ is the second member, and the whole is read thus: $2x$ plus $3y$ equals $a$ minus $5b$ plus $c$; which means that the numerical value of the first member is the same as the numerical value of the second member,—thus, $3 \times 4 + 2 \times 5 = 10 - 4 + 16$, or, $22 = 22$.

**16.** The sign $>$ or $<$ indicates that the quantities between which it is placed are *unequal,* the quantity on the side of the opening being the larger. Thus, $x > y$ indicates that $x$ is greater than $y$; also, $x < y$ indicates that $x$ is less than $y$.

1. The whole expression of which the sign $<$ or $>$ forms a part is called an INEQUATION. Thus, $2x + 5y > a - b + 2d$, is an inequation of which the first member is greater than the second.

**17.** The Signs of Proportion are thus written, : :: :, and $a : b :: c : d$, is read *a is to b, as c is to d.*

**18.** The sign $\propto$ indicates that one quantity *varies* as another. Thus, $x \propto y$ signifies that $x$ *varies as* $y$.

**19.** An *algebraic expression* is one involving *letters* and *signs*.

1. A *Monomial* is an algebraic expression consisting of *one* term. Thus, $5x^2y$.

2. A *Binomial* consists of *two* terms. Thus, $5x^2y + 4bc$.

3. A *Trinomial* consists of *three* terms. Thus, $x + 2y - 4c$.

4. A *Polynomial* consists of *many* terms. Thus, $x + 4y - 3z + 6$.

5. The *terms* of a polynomial are separated by the signs $+$ or $-$.

6. A monomial is *positive* or *negative* according as the sign is $+$ or $-$.

**20.** SIMILAR TERMS are such as have *like letters* and *exponents*. Thus, $3x^2y$ and $2x^2y$ are *similar;* but $3x^2y$ and $2xy^2$ are *dissimilar*.

**21.** A polynomial is *homogeneous* when the sum of the exponents in all the terms is the same. Thus, $4x^2y^3 + 5xy^4 - x^4z + 4xymnp$ is homogeneous, since the sum of the exponents in each term is 5.

**22.** The *reciprocal* of a quantity is 1 divided by the quantity. Thus, $\frac{1}{x}$ is the reciprocal of $x$.

1. The reciprocal of a *fraction* is the fraction *inverted*. Thus, the reciprocal of $\frac{x}{y}$ is $\frac{y}{x}$.

**23.** The sign $\therefore$ is the same as the words *therefore*, *hence*, or *consequently*.

**24.** The sign $\because$ is the same as the word *because*.

**25.** The letters of a term are usually written *alphabetically*, though this order is not essential. Thus, $3abc$ is the same as $3bca$.

**26.** The terms of a polynomial are usually arranged with reference to the exponent of the *leading letter*. Thus, $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$, where $x$ is considered the leading letter.

1. Of two polynomials involving the same letters, that is said to be *algebraically the greater* whose leading letter has the greater exponent. Thus, $x^3 - 3x^2y + 3xy^2 - y^3$ is greater than $x^2 - 2xy + y^2$.

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**27.** EXAMPLES.

Involving the Preceding Definitions.

1. Convert into algebraic language the square root of seven $a$ square, added to five $a$ multiplied by $m$.

*Ans.* $\sqrt{7a^2 + 5am}$, or $(7a^2 + 5am)^{\frac{1}{2}}$.

2. Convert into algebraic language three times the cube root of $x$ square, diminished by the square root of five $m$, multiplied by $n$ square, increased by twice the fifth root of $x$.

*Ans.* $3x^{\frac{2}{3}} - (5mn^2 + 2x^{\frac{1}{5}})^{\frac{1}{2}}$, or, $3x^{\frac{2}{3}} - \sqrt{5mn^2 + 2\sqrt[5]{x}}$.

3. Convert into algebraic language the fifth root of the sum of $x$ and $y$. *Ans.*

4. Convert into algebraic language the square root of $x$ increased by the cube root of $x$ square and the square root of $x$ cube. *Ans.* $\sqrt{x} + \sqrt[3]{x^2} + \sqrt{x^3}$, or, $x^{\frac{1}{2}} + x^{\frac{2}{3}} + x^{\frac{3}{2}}$.

5. Convert into common language the algebraic expression $3x^5 + (2x)^{\frac{1}{2}}$.

*Ans.* *Three times the fifth power of $x$ increased by the square root of two $x$.*

6. Write in common language the following algebraic expressions: $5x^4 - \sqrt{7x^3} + 12xy$, $x + \sqrt{x^{\frac{2}{3}} + \frac{1}{x^{\frac{3}{5}}}}$, and

$x + \sqrt{x^2 + \sqrt{x^3 + x^4 - x^5}}$.

7. Write in common language the following expressions:
$xy + \sqrt{4x^2 - \sqrt{5x^3 + 3x}}$., $\frac{x^2}{y^3} = 7x^2 + 8am$., $\frac{x+y}{x-y} > \sqrt[3]{2a - 5b}$.

8. Write in common language the following expressions:
$\frac{c}{2} < \frac{c^2 + a^2 - b^2}{2c}$, and $[x + (n-p)^{\frac{1}{2}} - \frac{2}{3}(a^2 - 5)^{\frac{3}{4}}]^{\frac{1}{5}}$.

REMARK.—The great advantage of algebraic symbols has been seen in the previous examples. By them are obtained both brevity and perspicuity.

**28.** The NUMERICAL VALUE of an algebraic expression is the value found by *arithmetical reduction* on affixing a numerical value to each of the letters composing the expression.

### EXAMPLES.

1. What is the numerical value of the expression $x + 2x^2$ when $x = 5$. *Ans.* $5 + 2 \times 5^2$, which is 55.

2. What is the value of the expression $5ab + 3a^2c - mn$ when $a = 2$, $b = 3$, $c = 4$, $m = 5$, and $n = 6$.
*Ans.* $5.2.3 + 3.2^2.4 - 5.6. = 48$.

3. Find the value of $a^{\frac{2}{3}} + b^{\frac{3}{5}} + c^{\frac{5}{2}}$ when $a = 8$, $b = 32$, $c = 4$, (*Vide*, **13**, 4.) *Ans.* $8^{\frac{2}{3}} + 32^{\frac{3}{5}} + 4^{\frac{5}{2}} = 4 + 8 + 32 = 44$.

4. Find the value of $x^{\frac{3}{2}} + 5x^{\frac{1}{3}} + x^{\frac{1}{6}}$ when $x = 64$. *Ans.* 534.

5. Find the value of $(x^{\frac{3}{2}} + y^{\frac{2}{3}} + z^{\frac{1}{4}}) . m$, when $x = 4$, $y = 8$, $z = 16$, $m = 2$. *Ans.* 28.

6. Find the value of $(x + y)(x - y)$ when $x = 3$, $y = 3$.
*Ans.* 0.

7. Find the value of $(x + y)(x + y)$ when $x = 4$, $y = 3$.
*Ans.* 49.

8. Find the value of $(x + y)(a + b)(x^2 - y^2)$ when $x = 4$, $y = 4$, $a = 2$, $b = 3$. *Ans.* 0.

9. Find the value of $(x + y + 3a + 2b)(x^3 - y^3)$ when $x = 2$, $y = 1$, $a = 4$, $b = 3$. *Ans.* 147.

10. Find the value of $\dfrac{abc}{ab + ac + bc}$ when $a = 1$, $b = 2$, $c = 3$.

*Ans.* $\dfrac{1.2.3}{1.2 + 1.3 + 2.3} = \dfrac{6}{11}$.

11. Find the value of $\dfrac{c^2 + b^2 - a^2}{2c}$ when $a = 50$, $b = 50$, $c = 40$.

*Ans.* 20.

12. Find the value of $x$ and $y$ in the equations $x = \dfrac{(a + b).c}{a}$, and $y = \dfrac{(a + b).c}{b}$, $c = 20$, $a = 1$, $b = 2$.

*Ans.* $x = 60$, $y = 30$.

13. Find the value of $x$ and $y$ in the equations, $x = \dfrac{(a + b) \times md - (c + d) \times nb}{ad - bc}$, and $y = \dfrac{(c + d) \times an - (a + b) \times cm}{ad - bc}$, when $a = 2$, $b = 1$, $m = 78$, $c = 7$, $d = 2$, $n = 79$.

*Ans.* $x = 81$, $y = 72$.

14. Find the value of $x$ in the equation $x = \frac{a}{2}\,(b + \sqrt{b^2 - 4^2})$, if $a = 4$, $b = 5$.

*Ans.* $x = 16$.

15. Find the value of $x$ and $y$ in the equations, $x = \dfrac{m^{\frac{1}{2}}\,n^{\frac{1}{2}}}{m^{\frac{1}{2}} - n^{\frac{1}{2}}}$ and $y = \dfrac{m^{\frac{1}{2}} + n^{\frac{1}{2}}}{m^{\frac{1}{2}}\,n^{\frac{1}{2}}}$ when $m = 9$, $n = 4$.

*Ans.* $x = 6$, $y = \frac{5}{6}$.

16. Find $x$ in the equation $x = \dfrac{n}{n^2 - m^2}(bn - \sqrt{a^2m^2 + b^2m^2 - a^2n^2})$, when $a = 2$, $b = 3$, $m = 5$, $n = 3$.

*Ans.* $x = 1\frac{1}{2}$.

17. Find $x$ and $y$ in the equations $x = \dfrac{(dn - bm) \times ac}{ac - bc}$ and $y = \dfrac{(am - nc)bd}{ad - bc}$, when $a = 5$, $b = 3$, $c = 4$, $d = 6$, $m = 1$, $n = 1$.

18. Find $x$ and $y$ in the equations $x = \dfrac{(a + b)c}{a}$ and $y = \dfrac{(a + b)c}{b}$, when $c = 25$, $a = 2$, $b = 3$.

19. Find $x$ and $y$ in the equations $x = \frac{1}{2}(b + \sqrt{\dfrac{4a - b^3}{3b}}$ and $y = \frac{1}{2}(b - \sqrt{\dfrac{4a - b^3}{3b}}$, when $a = 9$, $b = 3$.

20. Find the value of the expression $\frac{x^{-2}+y^{-3}}{x^{-2}-y^{-3}}$, when $x=2$, $y=8$.

21. Find the value of the expression $\sqrt{\frac{x^{\frac{3}{4}}-y^{\frac{2}{3}}}{y^{-\frac{1}{3}}-x^{-\frac{1}{2}}}}$ when $x=16$, $y=8$.

22. Find the value of the expression $15x^{a+1}y^{b+1}z^{c+1}$, when $x=2$, $y=2$, $z=2$, and $a=1$, $b=1$, $c=1$.

23. Find the value of the expression $(a+x)^0+2x^0y$, when $x=5$, $y=7$, $a=3$.

24. Find $x$ in the equation $x=\frac{1}{2}(b+\sqrt{-3b^2+2\sqrt{2(a+b^4)}})$, when $a=17$, $b=3$. *Ans.* $x=2$.

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## NOTATION.

**29.** The fundamental law of numeration in Arithmetic is this: *Figures increase from right toward the left in a tenfold ratio.*

If we wish to express a *number* of more than one figure by means of *letters* this *law* must be observed. Thus:

A number between 0 and 10 is expressed by any letter; as, $z$.

A number between 10 and 100 by *two* letters. Thus, $10x+y$.

A number between 100 and 1,000 by *three* letters. Thus, $100x+10y+z$.

This may be continued to any extent.

1. Find the value of the expression $10x+y$, when $x=2$, $y=1$; $x=4$, $y=5$, etc.

2. Find the value of the expression $100x+10y+z$, when $x=4$, $y=0$, $z=7$, etc.

3. Find the value of the expression $xy$, when $x=2$, $y=1$. *Ans.* 2.

4. Find the value of the expression $xyz$, when $x=4$, $y=0$, $z=7$. *Ans.* 0.

# CHAPTER II.

## ADDITION—SUBTRACTION—MULTIPLICATION—DIVISION.

## ADDITION.

**29.** ADDITION, in Algebra, consists in finding the simplest expression for the *sum* of several given expressions.

**30.** By Definition 14, we have $a + a = 2a$, $a + a + a = 3a$, $\therefore 2a + 3a = a + a + a + a + a = 5a$. Hence, to add similar positive monomials,

*Add the coëfficients and annex the common letters.*

EXAMPLES.

| | (1.) | (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|---|---|
| Add | $3a$ | $6x$ | $3ax$ | $3a^2x^{\frac{1}{2}}$ | $4a^3x^5$ |
| to | $5a$ | $11x$ | $7ax$ | $5a^2x^{\frac{1}{2}}$ | $5a^3x^5$ |
| *Ans.* | $8a$ | $17x$ | $10ax$ | $8a^2x^{\frac{1}{2}}$ | $9a^3x^5$ |

6. Add together $4x$, $5x$, $6x$, $9x$ and $25x$. *Ans.* $49x$.

7. Add together $7a^2x^4y$, $3a^2x^4y$, $9a^2x^4y$ and $a^2x^4y$.
*Ans.* $20a^2x^4y$.

**31.** By Definition 14, we have $-a - a = -2a$, $-a - a - a = -3a$, $\therefore -2a - 3a = -a - a - a - a - a = -5a$. Hence, to add similar negative monomials,

*Add the coëfficients, annex the common letters, and to the result prefix the sign* $-$.

EXAMPLES.

| | (1.) | (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|---|---|
| Add | $-3a$ | $-8a$ | $-10a^3x^2$ | $-5a^2x^{\frac{1}{3}}$ | $-11a^{\frac{1}{3}}x^{\frac{1}{5}}$ |
| to | $-5a$ | $-5a$ | $-10a^3x^2$ | $-13a^2x^{\frac{1}{3}}$ | $-20a^{\frac{1}{3}}x^{\frac{1}{5}}$ |
| *Ans.* | $-8a$ | $-13a$ | $-20a^3x^2$ | $-18a^2x^{\frac{1}{3}}$ | $-31a^{\frac{1}{3}}x^{\frac{1}{5}}$ |

6. Add together $-5a^2$, $-3a^2$, $-4a^2$, $-7a^2$, $-10a^2$ and $-a^2$. *Ans.* $-30a^2$.

7. Add together $-3x^2y$, $-4x^2y$, $-7x^2y$, $-9x^2y$ and $-x^2y$. *Ans.* $-24x^2y$.

**32.** When the monomials are not similar,

*Write the terms after each other, retaining the given signs.*

EXAMPLES.

| | (1.) | (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|---|---|
| Add | $a$ | $3a$ | $2a^2$ | $5x^3$ | $-4x^{\frac{1}{3}}$ |
| to | $b$ | $2x$ | $3a$ | $-3x^2$ | $3x^3$ |
| *Ans.* | $a+b$ | $3a+2x$ | $2a^2+3a$ | $5x^3-3x^2$ | $-4x^{\frac{1}{3}}+3x^3$ |

6. Add together $a$, $-b$, $c$ and $-x$. *Ans.* $a-b+c-x$.

**33.** To add similar monomials with unlike signs,

*Find the sum of the positive monomials by* 30.

*Find the sum of the negative monomials by* 31.

*Take the smaller coëfficient from the larger and annex the common letters.*

*Prefix the sign of the larger coëfficient to the result.*

EXAMPLES.

| | (1.) | (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|---|---|
| Add | $5a$ | $4x^2$ | $-7x$ | $-20x^5$ | $9x^{\frac{1}{2}}$ |
| to | $-2a$ | $-x^2$ | $3x$ | $11x^5$ | $-12x^{\frac{1}{2}}$ |
| *Ans.* | $3a$ | $3x^2$ | $-4x$ | $-9x^5$ | $-3x^{\frac{1}{2}}$ |

Add together

| (6.) | (7.) | (8.) | (9.) | (10.) |
|---|---|---|---|---|
| $5a$ | $5xy$ | $3x^2$ | $2x^{\frac{1}{3}}$ | $4xyz$ |
| $-3a$ | $-10xy$ | $5x^2$ | $-5x^{\frac{1}{3}}$ | $5xyz$ |
| $7a$ | $-13xy$ | $-4x^2$ | $-3x^{\frac{1}{3}}$ | $-9xyz$ |
| $8a$ | $8xy$ | $10x^2$ | $-7x^{\frac{1}{3}}$ | $3xyz$ |
| $12a$ | $14xy$ | $-12x^2$ | $-10x^{\frac{1}{3}}$ | $-7xyz$ |
| $-13a$ | $-20xy$ | $7x^2$ | $8x^{\frac{1}{3}}$ | $-10xyz$ |
| *Ans.* $16a$ | $-16xy$ | $9x^2$ | $-15x^{\frac{1}{3}}$ | $-14xyz$ |

11. Add together $a^2x$, $5a^2x$, $-7a^2x$, $13a^2x$, $-9a^2x$, $-20a^2x$, and $-3a^2x$. *Ans.* $-20a^2x$.

12. Add together 5, $-4$, 17, $-30$, and 80. *Ans.* 68.

13. Add together $7x^2y^2$, $-9x^2y^2$, $4x^2y^2$, $-x^2y^2$. *Ans.* $x^2y^2$.

14. Add together $4xy^2$, $7xy^2$, $9xy^2$, $-17xy^2$. *Ans.* $3xy^2$.

15. Add together $15x^2y$, $16x^2y$, $18x^2y$, $-34x^2y$.

16. Add together $-12x^4y^2$, $-20x^4y^2$, and $30x^4y^2$.

17. Add together $-19x^2y^4$, $-21x^2y^4$, and $45x^2y^4$.

18. Add together $-30x^2y^3$, $40x^2y^3$, and $18x^2y^3$.

19. Add together $4x$, $5x$, $-3x$, and $-6x$. *Ans.* 0.

20. Add together $5xyz$, $7xyz$, and $-12xyz$.

**34.** To add polynomials having in each similar terms,

*Arrange the polynomials so that similar terms stand under each other.*

*Add each column of terms by* **33.**

EXAMPLES.

| | (1.) | (2.) | (3.) |
|---|---|---|---|
| Add | $x + y$ | $3x + 2y + z$ | $7x - 5y + 6z - 10$ |
| to | $x - 2y$ | $5x - 4y - 6z$ | $4x + 10y - 9z + 30$ |
| | $2x - y$ | $8x - 2y - 5z$ | $11x + 5y - 3z + 20$ |

| | (4.) | (5.) | (6.) |
|---|---|---|---|
| Add | $4x+3y-2z$ | $3x^2+2y^3-4z^4+10$ | $x^4+x^2y+5xy^2$ |
| | $-5x+4y+6z$ | $4x^2-2y^3+5z^4-10$ | $3x^4-2x^2y+7xy^2$ |
| | $7x-8y-9z$ | $-5x^2+3y^3-2z^4+15$ | $-9x^4-4x^2y-13xy^2$ |
| | $4x-y+z$ | $6x^2-8y^3+4z^4-16$ | $7x^4+3x^2y+xy^2$ |
| *Ans.* | $10x-2y-4z$ | $8x^2-5y^3+3z^4-1$ | $2x^4-2x^2y$ |

7. Add together the polynomials $3a+2b-5c+12x-10$, $-7a+3b-6c-13x+12$, $4a+3b-10c-5x+8$, and $10a-7b+3c-x+13$.

*Solution.*

$$\begin{array}{r}
3a+2b-5c+12x-10\\
-7a+3b-6c-13x+12\\
4a+3b-10c-5x+8\\
10a-7b+3c-x+13\\
\hline
10a+b-18c-7x+23
\end{array}$$

8. Add together the polynomials $7a^2x+5b^3-7m^{\frac{1}{2}}+14n$, $3a^2x-7n+9b^3-5m^{\frac{1}{2}}$, $-10b^3+4m^{\frac{1}{2}}+8n-10a^2x$.

*Solution.*

$$\begin{array}{r}
7a^2x+5b^3-7m^{\frac{1}{2}}+14n\\
3a^2x+9b^3-5m^{\frac{1}{2}}-7n\\
-10a^2x-10b^3+4m^{\frac{1}{2}}+8n\\
\hline
4b^3-8m^{\frac{1}{2}}+15n
\end{array}$$

9. Add the polynomials $2x+3y-z$ and $2x-3y+z$.

*Ans.* $4x$.

10. Add together $x+2y+c$ and $-x+2y-c$.

*Ans.* $4y$.

11. Add together $x+8y-5z+g+9$, $6y-2x+3z-1-g$, $7g+1-3z+y-x$, $-g-8-3y-x$, and $3+5z-9y-g+7x$. *Ans.* $4x+3y+5g+4$.

12. Add together $3x^{\frac{1}{2}}y + 2x^2y^3 - 4x^2y + 3xy^3$, $4x^2y - 3xy^3 - 2x^2y^3 + 2x^{\frac{1}{2}}y$, $-10x^2y + 8xy^3 + 5x^2y^3 + 7x^{\frac{1}{2}}y$ and $10x^2y - 8xy^3 - 5x^2y^3 - 5x^{\frac{1}{2}}y$. *Ans.* $7x^{\frac{1}{2}}y$.

13. Add together $2xy + 7x^2y - 8xy^2 + 3x^{\frac{1}{2}}y^{\frac{1}{2}}$, $8xy^2 - 5xy^2 + 2xy^2 - 3x^{\frac{1}{2}}y^{\frac{1}{2}}$, $15x^{\frac{1}{2}}y^{\frac{1}{2}} - 18x^{\frac{1}{2}}y^{\frac{1}{2}} - 3x^2y + 7xy$ and $-2xy - 7x^2y + 4xy - 8x^2y$. *Ans.* $11xy - 11x^2y - 3xy^2 - 3x^{\frac{1}{2}}y^{\frac{1}{2}}$.

14. Add together $3x^{\frac{1}{2}}y^{\frac{1}{3}} - 2xyz + 3x$, $7xyz - 3x - 5x^{\frac{1}{2}}y^{\frac{1}{3}}$ and $-5xyz + 2x^{\frac{1}{2}}y^{\frac{1}{3}}$. *Ans.* 0.

15. Add together $x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{\frac{1}{4}} - x^{\frac{1}{5}}$, $5x^{\frac{1}{2}} + 7x^{\frac{1}{3}} - 8x^{\frac{1}{4}} + 5x^{\frac{1}{5}}$ and $-2x^{\frac{1}{3}} + 3x^{\frac{1}{2}} + 7x^{\frac{1}{4}} - 10x^{\frac{1}{5}}$. *Ans.* $9x^{\frac{1}{2}} + 6x^{\frac{1}{3}} - 6x^{\frac{1}{5}}$.

16. Add together $3xy + 2a^2b^2 - 5mn + 40$, $3mn - 20 - 5a^2b^2 + 2xy$ and $2mn - 10 + 3a^2b^2 - 7xy$.

17. Add together $x^{\frac{3}{2}} + y^{\frac{5}{3}} + xy + x^2y^2 + x$, $-7x - 4x^2y^2 - 3xy - 3y^{\frac{5}{3}} + 2x^{\frac{3}{2}}$ and $4xy - 7x^2y^2 + 3x - 5y^{\frac{5}{3}} + 3x^{\frac{3}{2}}$.

18. Add together $3x^4 - c + 7m - n + 10$, $-5x^4 - 2c - 7m + n - 15$, $7x^4 - 2c + 3m - 2n$ and $-10x^4 - 5c + 5m + 3x^4 - 3c + 2x^4$.

19. Add together $4x^2y + 5x + 3m + y + 2pq + 4z$, $3x^2y - 10x + 4m - 7y + 5pq - 8z$ and $-7m + 7y - 7pq + 5z + 6x - 7x^2y$.

20. Add together $x^2 + 3x^2m + 5y^2 + 4p + 3z^2$, $-10x^2m - 7y^2 + 5p - 10z^2 + 5x^2$ and $8z^2 - 9p + 3y^2 + 7x^2m - 5x^2$.

21. Add together $3x^2 + 2xy + y^2$, $-2xy + 3y^2 + 5x^2$ and $-4y^2 + 4xy + 2x^2$.

22. Add together $x^2 + 2xy + y^2$ and $x^2 - 2xy + y^2$.

23. Add together $x^3 + 3x^2y + 3xy^2 + y^3$ and $x^3 - 3x^2y + 3xy^2 - y^3$.

24. Add together $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ and $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

25. Add together $x^2 + xy + y^2$ and $x^2 - xy + y^2$.

26. Find the numerical value of the last five examples when $x=2$ and $y=2$. *Ans.* $21^{st}=56$, $22^{nd}=16$, $23^{rd}=64$, $24^{th}=256$, $25^{th}=16$.

27. Add together $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ and $x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

28. Add together $x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$ and $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$.

29. Find the numerical value of the last two examples, when $x=1$, $y=2$. *Ans.* 6 and 730.

30. Add together $3(x+y)$, $2(x+y)$ and $8(x+y)$.

*Ans.* $13(x+y)$.

31. Add together $2(x^2+y^3)+5(x+y+z)+4(x^3+2y^2)$ and $6(x^2+y^3)-4(x+y+z)-2(x^3+2y^2)$, where $x=1$, $y=2$, $z=3$.

*Ans.* $8(x^2+y^3)+(x+y+z)+2(x^3+2y^2)=96$.

32. Add together $x+l(x+y+z)+4$, $y+m(x+y+z)-3$ and $z+n(x+y+z)+5$, and find the numerical value when $x=1$, $y=2$, $z=3$, $l=4$, $m=5$, $n=6$.

*Ans.* $x+y+z+(l+m+n)(x+y+z)+6=102$.

33. Add together $3x(1+2y)+9$, $5x(1+2y)-7$, $-2\frac{1}{2}x(1+2y)+12$ and $7\frac{1}{2}x(1+2y)-8$. *Ans.* $13x(1+2y)+6$.

34. Add together $7+\frac{1}{2}(2c+d-m)+3x^2$, $-8+\frac{1}{3}(2c+d-m)-5x^2$ and $8-\frac{1}{6}(2c+d-m)+2x^2$.

*Ans.* $7+\frac{2}{3}(2c+d-m)$.

35. Add together $5(x+y)^2+7$, $4(x+y)^2-3$ and $-8(x+y)^2-4$. *Ans.* $(x+y)^2$.

36. Add together $x+y$ and $x-y$. *Ans.* $2x$.

**35.** By the last example we see that

*The sum of two numbers added to their difference gives twice the larger number.*

EXAMPLES.

1. $(12 + 5) + (12 - 5) = 2 \times 12 = 24.$
2. $(4\frac{1}{2} + 3) + (4\frac{1}{2} - 3) = 2 \times 4\frac{1}{2} = 9.$
3. $(6 + 2\frac{1}{4}) + (6 - 2\frac{1}{4}) = 12.$
4. $(17 - 3\frac{1}{2}) + (17 + 3\frac{1}{2}) = 34.$
5. $(8\frac{1}{2} - 2\frac{1}{4}) + (8\frac{1}{2} + 2\frac{1}{4}) = 17.$
6. $(3\frac{1}{8} + 1\frac{5}{6}) + (3\frac{1}{8} - 1\frac{5}{6}) = 6\frac{1}{4}.$
7. $(4 - 2) + (4 + 2) =$
8. $(8x + 2y) + (8x - 2y) = 16x.$
9. $(2\frac{1}{2}x + 3\frac{1}{3}y) + (2\frac{1}{2}x - 3\frac{1}{3}y) =$
10. $(5x^4 + 2y) + (5x^4 - 2y) =$
11. What is the value of the last three examples when $x = 5$, $y = 3$? *Ans.* $8^{th} = 80$, $9^{th} = 25$, $10^{th} = 6250$.

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# SUBTRACTION.

**36.** Subtraction in Algebra consists in finding the *simplest expression* for the *difference* of *two given expressions*, or, in finding what quantity added to the *subtrahend* will produce the *minuend.*

**37.** Of the two given expressions that which is to be subtracted is called the *subtrahend;* the other is called the *minuend.*

**38.** To find the difference of two similar monomials:

This difference may always be expressed by either a positive or a negative quantity, each result depending upon which of the given expressions is taken for the minuend; thus,

*Vide* **24, 36.**

1. From $5a$ subtract $3a$ and we have $2a \because 3a + 2a = 5a.$
2. From $3a$ subtract $5a$ and we have $-2a \because 5a + (-2a) = 3a.$

3. From $5a$ subtract $-3a$ and we have $8a$ $\because$ $-3a+8a=5a$.
4. From $-3a$ subtract $5a$ and we have $-8a$ $\because$ $5a+(-8a) = -3a$.

5. From $-5a$ subtract $3a$ and we have $-8a$ $\because$ $3a+(-8a) = -5a$.
6. From $3a$ subtract $-5a$ and we have $8a$ $\because$ $-5a+8a=3a$.

7. From $-5a$ subtract $-3a$ and we have $-2a$ $\because$ $-3a+(-5a) = -8a$.
8. From $-3a$ subtract $-5a$ and we have $2a$ $\because$ $-5a+2a = -3a$.

These may be arranged as follows:

| | (1.) | (2.) | (3.) | (4.) | (5.) | (6.) | (7.) | (8.) |
|---|---|---|---|---|---|---|---|---|
| From | $5a$ | $3a$ | $5a$ | $-3a$ | $-5a$ | $3a$ | $-5a$ | $-3a$ |
| take | $3a$ | $5a$ | $-3a$ | $5a$ | $3a$ | $-5a$ | $-3a$ | $-5a$ |
| *Ans.* | $2a$ | $-2a$ | $8a$ | $-8a$ | $-8a$ | $8a$ | $-2a$ | $2a$ |

By comparing (1) and (2) it will be seen that the difference between $5a$ and $3a$ is expressed either by $2a$ or by $-2a$.

There is a similar relation between (3) and (4), (5) and (6), (7) and (8).

Each of these results may be obtained by the following rule:

*Consider the sign of the subtrahend changed.*

*If the signs are then alike, add the coëfficients, prefix the common sign, and annex the common letters.*

*If the signs are unlike, subtract the less coëfficient from the greater, prefix the sign of the larger coëfficient, and annex the common letters.*

EXAMPLES.

| | (1.) | (2.) | (3.) | (4.) | (5.) | (6.) | (7.) | (8.) |
|---|---|---|---|---|---|---|---|---|
| From | $10x$ | $10x$ | $-10x$ | $-10x$ | $3x$ | $-3x$ | $3x$ | $-3x$ |
| take | $3x$ | $-3x$ | $3x$ | $-3x$ | $10x$ | $10x$ | $-10x$ | $-10x$ |
| *Ans.* | $7x$ | $13x$ | $-13x$ | $-7x$ | $-7x$ | $-13x$ | $13x$ | $7x$ |

| | (9.) | (10.) | (11.) | (12.) |
|---|---|---|---|---|
| From | $5a^2x$ | $15a^{\frac{1}{2}}x^{\frac{3}{4}}y$ | $12axyz$ | $-4x^5$ |
| take | $7a^2x$ | $-16a^{\frac{1}{2}}x^{\frac{3}{4}}y$ | $30axyz$ | $7x^5$ |
| *Ans.* | $-2a^2x$ | $31a^{\frac{1}{2}}x^{\frac{3}{4}}y$ | $-18axyz$ | $-11x^5$ |

13\. From $7a^2x$ subtract $5a^2x$. *Ans.*

14\. From $-16a^{\frac{1}{2}}x^{\frac{3}{4}}y$ subtract $15a^{\frac{1}{2}}x^{\frac{3}{4}}y$. *Ans.*

15\. From $30axyz$ subtract $-12axyz$. *Ans.*

16\. From $7x^5$ subtract $-4x^5$. *Ans.*

17\. From $10x^2$ subtract $10x^2$. *Ans.* 0.

**39.** To find the difference of two monomials not similar:

*Change the sign of the subtrahend and write it after the minuend.*

EXAMPLES.

| | (1.) | (2.) | (3.) | (4.) | (5.) | (6.) |
|---|---|---|---|---|---|---|
| From | $a$ | $5a$ | $3x$ | $7y^2$ | $4xy$ | $-12x^2$ |
| take | $b$ | $-2b$ | $4y$ | $4m$ | $3x$ | $-3x$ |
| *Ans.* | $a-b$ | $5a+2b$ | $3x-4y$ | $7y^2-4m$ | $4xy-3x$ | $-12x^2+3x$ |

7\. From $3x$ subtract $2y$.

8\. From $7x^3$ subtract $2x^2$.

9\. From $10x$ subtract $-4m$.

10\. From $5xy$ subtract 12.

11\. From 12 subtract $-4x$.

12\. From 15 subtract $-2x^{\frac{1}{4}}$.

**40.** To find the difference of two polynomials:

*Write the polynomial taken as the subtrahend under that taken as the minuend, placing similar terms under each other and those not similar in any order.*

*Subtract similar terms by the rule in* **38.**

*Subtract terms not similar by* **39.**

EXAMPLES.

| | (1.) | (2.) | (3.) | (4.) |
|---|---|---|---|---|
| From | $3x - y$ | $x - y$ | $x^2 + 2xy + y^2$ | $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$ |
| take | $2x - b$ | $x + 2y$ | $x^2 - 2xy + y^2$ | $x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y$ |
| *Ans.* | $x - y + b$ | $-3y$ | $4xy$ | $2x^{\frac{1}{2}}y^{\frac{1}{2}}$ |

5. From $3a + 2b - 3c + 4g - 5m + 2n - 7x - y + 10$
   take $a - 3b + 4c - 2g + 5m - n + 4x - y - 40$
   *Ans.* $2a + 5b - 7c + 6g - 10m + 3n - 11x \quad + 50$

| | (6.) | (7.) | (8.) |
|---|---|---|---|
| From | $x^2 + 3xy + y^2$ | $x^2 + xy + y^2$ | $x$ |
| take | $-x^3y + 3xy + xy^3$ | $-x^2 + xy - y^2$ | $x - y$ |
| *Ans.* | $x^2 + x^3y + y^2 - xy^3$ | $2x^2 \quad + \quad 2y^2$ | $y$ |

9. From $x - y$ subtract $x - 2y$. *Ans.* $y$.

10. From $x + y$ subtract $x$. *Ans.* $y$.

11. From $x^2 - xy + y^2$ subtract $x^2 - xy - y^2$. *Ans.* $2y^2$

12. From $x + \sqrt{xy} + y$ subtract $x - \sqrt{xy} + y$. *Ans.* $2\sqrt{xy}$.

13. From $x^3 + 3x^2y + 3xy^2 + y^3$ subtract $x^3 - 3x^2y + 3xy^2 - y^3$.
*Ans.* $6x^2y + 2y^3$.

14. From $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ subtract $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

15. From $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.
subtract $x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$.
*Ans.* $12x^5y \quad + 40x^3y^3 \quad + 12xy^5$

16. From $3m + 2n - 5xy + 4p - q + 2x$ subtract $2m - 3n - 4p + 5xy + 2x - q$.

17. From $x^{\frac{1}{2}}y^{\frac{1}{2}} + xy + x^2y^2$ subtract $x^{\frac{1}{2}}y^{\frac{1}{2}} - xy + x^2y^2$.

18. From $x^6 + 3x^4y^2 + 3x^2y^4 + y^6$ subtract $-x^5y - 3x^4y^2 + 3x^2y^4 + x^6$.

19. From $x^6 + 3x^4y^2 + 3x^2y^4 + y^6$ subtract $x^5y + 3x^4y^2 - 3x^2y^4 + x^6$.

20. From $x + 2y - 4$ subtract $x - 3y + 7$.

21. From $4x + 7y - 6$ subtract $2x + 7y + 12$.

22. From $x^{\frac{3}{2}} + 5x^{\frac{2}{3}} - 7x^{\frac{1}{2}}$ subtract $2x^{\frac{3}{2}} - 3x^{\frac{2}{3}}$.

23. From $1 + x + x^2 + x^3$ subtract $1 - x + x^2 - x^3$.

24. From $1 + 2x + 3x^2 + 5x^3$ subtract $1 - 2x + 3x^2 - 5x^3$.

25. From $1 + 5x + 6x^2$ subtract $1 - 5x + 6x^2$.

26. From $x + x^2 + x^3$ subtract $x - x^2 - x^3$.

27. From $x^2 + 2xy + y^2$ subtract $x^2 + xy$.

28. From $x^3 + y^3$ subtract $x^3 + x^2y$.

29. From $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$ subtract $x^6 - 3x^5y + 3x^4y^2 - x^3y^3$.

**41.** From the sum of two or more quantities to subtract any number of quantities:

*Change the signs of the subtrahends and add the columns as in* **33.**

EXAMPLES.

1. From the sum of $4x + 3y - 2z$ and $-5x + 4y + 6z$ take $-7x + 8y + 9z$ and $-4x + y - z$. (*Vide* **34**, ex. 4.)

*Ans.* $10x - 2y - 4z$.

2. From the sum of $3x^2 + 2y^3 - 4z^4 + 10$ and $4x^2 - 2y^3 + 5z^4 - 10$ take $5x^2 - 3y^3 + 2z^4 - 15$ and $-6x^2 + 8y^3 - 4z^4 + 16$.

*Ans.* $8x^2 - 5y^3 + 3z^4 - 1$.

3. From the sum of $5x - 4xy + 8y^2$ and $6xy - 7x + 4y^2 + 8$ take $3x + 2xy - 5y^2 + 4$ and $5x + 2xy + 10y^2 - 15$.

4. From $x^2 + 2xy + y^2$, $x^2 - 2xy + y^2$ and $x^2 + xy + 5y^2$ take $x^2 - 5xy - 2y^2$, $2x^2 + 4xy + 9y^2$ and $-x^2 + 2xy$.

5. From $2(x - y) + z$, $3(x - y) + 2z$ and $5(x - y) - z$ take $4(x - y) + z - 2$.

6. From $3(x - y)^{-2} + 4$ and $2(x - y)^{-2} + 6$ take $4(x - y)^{-2} + 8$ and find the value when $x = 8$, $y = 3$.

*Ans. to last point* $2\frac{1}{25}$.

7. From $7(x+y)^{\frac{1}{2}}+15$ and $8(x+y)^{\frac{1}{2}}-16$ take $12(x+y)^{\frac{1}{2}}-1$ and find the value when $x=16$, $y=9$.

*Ans. to last point* 15.

8. From $8(x^2+y^2)^{\frac{1}{2}}+2xy$ and $4(x^2+y^2)^{\frac{1}{2}}+3xy$ take $10(x^2+y^2)^{\frac{1}{2}}+4xy$ and find the value when $x=4$, $y=3$.

*Ans.* 22.

**42.** In the expressions

$$+(+a), \qquad +(-a), \qquad -(+a) \quad \text{and} \quad -(-a)$$

the sign before the parenthesis is called the *sign of operation.*

The sign before the letter is called the *sign of the quantity.*

Thus,

(1.) $+(+a)$ means that the positive quantity $a$ is to be added.

(2.) $+(-a)$ means that the negative quantity $-a$ is to be added.

(3.) $-(+a)$ means that the positive quantity $a$ is to be subtracted.

(4.) $-(-a)$ means that the negative quantity $-a$ is to be subtracted.

By performing the operations indicated by the *sign of operation*, we have,

(1.) $+(+a)=+a$, (3.) $-(+a)=-a$,

(2.) $+(-a)=-a$, (4.) $-(-a)=+a$,

where the sign in the second member of each equation is called the *essential sign.*

(5.) By comparing (1) and (4) it is seen that the *addition* of a *positive* quantity is the same as the *subtraction* of an equal *negative* quantity; that is $+(+a)=-(-a)$.

(6.) By comparing (2) and (3) it is seen that the *addition* of a *negative* quantity is the same as the *subtraction* of an equal *positive* quantity; that is $+(-a)=-(+a)$.

(7.) It is plain, then, that $-(-ab)=+ab$.

EXAMPLES.

1. What is the value of $3x-(+5x)$? *Ans.* $-2x$.
2. What is the value of $3x+(-5x)$? *Ans.* $-2x$.
3. What is the value of $3x+(+5x)$? *Ans.* $8x$.
4. What is the value of $3x-(-5x)$? *Ans.* $8x$.

**43.** The subtraction of a polynomial is indicated by inclosing it in a parenthesis and prefixing the sign $-$. Thus, $x+y-(x-y)$ signifies that $x-y$ is to be subtracted from $x+y$.

Performing the operations we have $x+y-(x-y)=x+y-x+y=2y$. Hence, to remove a parenthesis having a *negative sign of operation*,

*Change all the signs in the parenthesis and unite the terms as in addition.*

EXAMPLES.

1. Remove the parenthesis from $a-(b+c)$. *Ans.* $a-b-c$.
2. Remove the parenthesis from $a-(b-c)$. *Ans.* $a-b+c$.
3. Remove the parenthesis from $a-(-b+c)$. *Ans.* $a+b-c$.
4. Remove the parenthesis from $a-(-b-c)$. *Ans.* $a+b+c$.
5. Remove the parenthesis from $x+2y-4-(x-3y+7)$. (*Vide* **40**, ex. 20). *Ans.* $5y-11$.
6. Remove the parentheses from $a-[b-(c-d)+x]$. *Ans.* $a-(b-c+d+x)=a-b+c-d-x$.
7. Remove the parentheses from $a-[[b-[c-(d-e)-f]-g]]$. *Ans.* $a-b+c-d+e-f+g$.

**44.** By reversing the operations of **43**, polynomials may be written in various ways. Thus,

1. $$x-5x^2+6x^3+7x^4-8x^5,$$

is the same as $$x-(5x^2-6x^3-7x^4+8x^5),$$

which is the same as $$x-[5x^2-(6x^3+7x^4)+8x^5].$$

What is the value of either polynomial when $x=2$? *Ans.* $-114$.

2. Find the value of $a - [[-b - [c - (-d - f) + g] - h]]$ when $a = 1$, $b = 2$, $c = 3$, $d = 4$, $f = 5$, $g = 6$ and $h = 7$.

**45.** Since $x + y - (x - y) = 2y$, it is plain that

*The difference of two numbers taken from their sum gives twice the smaller number.*

EXAMPLES.

1. $(12 + 5) - (12 - 5) = 2 \times 5 = 10.$
2. $(4\frac{1}{2} + 3) - (4\frac{1}{2} - 3) = 2 \times 3 = 6.$
3. $(6 + 2\frac{1}{2}) - (6 - 2\frac{1}{2}) =$ 5.
4. $(4 + 2) - (4 - 2) =$
5. $(17 + 3\frac{1}{2}) - (17 - 3\frac{1}{2}) =$ 7.
6. $(7 + 2\frac{1}{2}) - (7 - 2\frac{1}{2}) =$ 5.
7. $(10\frac{1}{2} + 5\frac{3}{4}) - (10\frac{1}{2} - 5\frac{3}{4}) =$ $11\frac{1}{2}$.
8. $(6\frac{1}{5} + 2\frac{3}{10}) - (6\frac{1}{5} - 2\frac{3}{10}) =$ $4\frac{3}{5}$.
9. $(3x^4 + 5y) - (3x^4 - 5y) =$
10. $(2x^{\frac{1}{2}} + 3y^{\frac{1}{3}}) - (2x^{\frac{1}{2}} - 3y^{\frac{1}{3}}) =$

What is the value of 9th and 10th examples when $x = 256$, $y = 8$?

**46.** PROBLEMS IN SUBTRACTION.

1. If A is worth $5a$ dollars and B $3a$ dollars, what is the difference between their pecuniary conditions?

*Ans.* $+2a$ or $-2a$ dollars.

2. If A is worth $5a$ dollars and B is in debt $3a$ dollars, what is the difference between their pecuniary conditions?

*Ans.* $+8a$ or $-8a$ dollars.

3. If A is in debt $5a$ dollars, and B is worth $3a$ dollars, what is the difference between their pecuniary conditions?

*Ans.* $-8a$ or $+8a$ dollars.

4. If A is in debt $5a$ dollars, and B is also in debt $3a$ dollars, what is the difference between their pecuniary condition?

*Ans.* $-2a$ or $+2a$ dollars.

If $a = \$3000$ what are the answers of each example?

**47.** By distinguishing what each one is worth by $+$ and what each one is in debt by $-$, the connection of these problems with the previous principles is evident, (*Vide* **38.**). It is seen that the difference between two quantities can be expressed as well by a negative as by a positive result. In Arithmetic it is not necessary to recognize this fact, but in Algebra it is of the utmost importance to have a correct apprehension of it.

**48.** Merely to find a difference it is of no consequence which of two given quantities we call the minuend or which the subtrahend. After having, however, assumed one of the quantities to be the minuend, *the result must always be referred to it.*

**49.** The difference between 7 and 4 is $+3$ or $-3$; thus,

$$\left.\begin{array}{r} 7 \\ \underline{4} \\ +3 \end{array}\right\} +3 \text{ showing that 7 is 3 \textit{greater} than 4.} \qquad \left.\begin{array}{r} 4 \\ \underline{7} \\ -3 \end{array}\right\} -3 \text{ showing that 4 is 3 \textit{less} than 7.}$$

**50.** The negative sign, then, serves to indicate some peculiar circumstance connected with the quantity before which it is placed.

**51.** In a given problem, *negative* quantities have a sense contrary to that which limits *positive* quantities.

## MULTIPLICATION.

**52.** MULTIPLICATION is the operation of finding the *product* of two quantities.

**53.** The *multiplicand* is the quantity to be multiplied.

**54.** The *multiplier* is the quantity by which to multiply.

**55.** The multiplicand and the multiplier may be interchanged at pleasure.

**56.** To multiply positive monomials; (*Vide* Def. **19**, 6.):

By Def. **10** we have $x \times y = xy$.

By Def. **13** we have $x.x = x^2$ and $x.x.x = x^3$ $\therefore$ $x^2 \times x^3 = xxxxx = x^5$.

By Def. **14** we have $5a \times 3 = 5a + 5a + 5a = 15a$ $\therefore$ $5ax^3 \times 3x^2y = 15ax^5y$.

Hence, (*vide* **25**,)

*Multiply the coëfficients and add the exponents of like letters.*

EXAMPLES.

| | (1.) | (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|---|---|
| Multiply | $7a^2b$ | $2a^2x$ | $16a^3b^4$ | $10a^{\frac{1}{2}}b^{\frac{1}{3}}$ | $7x^{\frac{2}{5}}y^{\frac{4}{7}}$ |
| by | $5ax$ | $5a^3x^2$ | $2ab$ | $2a^{\frac{1}{2}}b^{\frac{2}{3}}$ | $3x^{\frac{3}{5}}y^{\frac{3}{7}}$ |
| *Ans.* | $35a^3bx$ | $10a^5x^3$ | $32a^4b^5$ | $20ab$ | $21xy$ |

| | (6.) | (7.) | (8.) | (9.) | (10.) |
|---|---|---|---|---|---|
| Multiply | $\frac{3}{5}x^{\frac{1}{2}}y^{\frac{1}{3}}$ | $\frac{3}{7}xyz^{-4}$ | $8x^2y^{\frac{1}{3}}$ | $5x^{\frac{1}{3}}y^{\frac{3}{4}}$ | $3x^{\frac{1}{7}}y^{-5}$ |
| by | $\frac{5}{3}x^{\frac{1}{2}}y^{\frac{2}{3}}$ | $\frac{7}{5}x^{-1}yz^4$ | $7x^{\frac{1}{2}}y^{\frac{3}{4}}$ | $5x^{\frac{1}{3}}y^{\frac{1}{5}}$ | $5x^{\frac{1}{7}}y^4$ |
| *Ans.* | $x\ y$ | $\frac{3}{5}y^2$ | | | $15x^{\frac{2}{7}}y^{\ 1}$ |

**57.** To multiply two monomials with unlike signs:

By Def. **14** we have $-5a \times 3 = -5a - 5a - 5a = -15a$.

But $-5a \times 3$ is the same as $3 \times -5a$. (*Vide* **55.**)

Hence,

*Multiply as in* **56**, *and prefix the sign* — *to the product.*

EXAMPLES.

| | (1.) | (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|---|---|
| Multiply | $4a^3b$ | $-2x^{\frac{1}{2}}y^{\frac{1}{3}}$ | $-4m^{\frac{3}{5}}n^{\frac{4}{3}}$ | $-7x^{\frac{1}{7}}y^{14}$ | $2m^{\frac{1}{5}}x^{\frac{1}{7}}$ |
| by | $-2a^2b$ | $3x^{\frac{1}{2}}y^{\frac{2}{3}}$ | $2m^{\frac{7}{5}}n^{\frac{8}{3}}$ | $5x^{\frac{6}{7}}y^{20}$ | $-2m^{\frac{4}{5}}x^{\frac{6}{7}}$ |
| *Ans.* | $-8a^5b^2$ | $-6xy$ | $-8m^2n^4$ | | |

| | (6.) | (7.) | (8.) | (9.) |
|---|---|---|---|---|
| Multiply | $-2x^{-1}y^{-1}z^{-1}$ | $-4x^{-1}y^{-1}z^{-1}$ | $8x^2y^3z$ | $3xyz$ |
| by | $\frac{1}{2}x\,y\,z$ | $6x^2y^2z^2$ | $-3ax^2y$ | $-2abc$ |
| *Ans.* | $-1$ | | | |

**58.** To multiply negative monomials:

It is plain that $-a = -1(+a)$ and $-b = +1(-b)$.

Hence

| | | | |
|---|---|---|---|
| Multiply | $-a =$ | $-1(+a)$ | *Vide* Def. **14**, 3. |
| by | $-b =$ | $+1(-b)$ | |
| and we have | | $-1(-ab) = -(-ab) = ab.$ | *Vide* **42**, 7. |

Therefore

*Multiply as in* **56**, *and prefix the sign* $+$ *to the product.*

EXAMPLES.

| | (1.) | (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|---|---|
| Multiply | $-x$ | $-x$ | $-3a^2b$ | $-2a^{\frac{1}{2}}x^{\frac{1}{3}}y^{\frac{1}{4}}$ | $-5xyz$ |
| by | $-y$ | $-x$ | $-4a^2bx$ | $-\frac{1}{2}a^{\frac{1}{2}}x^{\frac{2}{3}}y^{\frac{3}{4}}$ | $-3xyz$ |
| *Ans.* | $xy$ | $x^2$ | | | |

(*Vide* Def. **9**, last clause.)

| | (6.) | (7.) | (8.) | (9.) |
|---|---|---|---|---|
| Multiply | $-3xy^2$ | $-5xyz$ | $-7xy$ | $-5a^2bc$ |
| by | $-4ay^2$ | $-8xyz^2$ | $-2ab$ | $-9axy$ |

**59.** Hence, to multiply algebraic monomials:

*Multiply the coëfficients, and add the exponents of like letters.*

*Like signs produce* $+$, *and unlike signs produce* $-$.

EXAMPLES.

| | (1.) | (2.) | (3.) |
|---|---|---|---|
| Multiply | $-5a\,x\,y\,z$ | $5a^x\,b^y\,c^x$ | $-5x^a\,y^b\,z^c$ |
| by | $4a\,x\,y\,z$ | $-3a^x\,b^{2y}\,c^{3z}$ | $-3x\,y\,z$ |
| *Ans.* | $-20a^2x^2y^2z^2$ | $-15a^{2x}b^{3y}c^{4z}$ | $15x^{a+1}y^{b+1}z^{c+1}$ |

4. Multiply $-10a^{\frac{1}{5}}x^3y^4$ by $4a^{\frac{4}{5}}x^{\frac{1}{2}}y^{\frac{1}{3}}$. *Ans.* $-40ax^{\frac{7}{2}}y^{\frac{13}{3}}$.
5. Multiply $7x^2y^2z^2$ by $3x^3y^3z^3$.
6. Multiply $11x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}}$ by $-2x^{\frac{3}{2}}y^{\frac{5}{3}}z^{\frac{7}{4}}$.
7. Multiply $-13mnp$ by $6mnp$.
8. Multiply $-26x^{\frac{1}{4}}y^{\frac{1}{5}}z^{\frac{1}{6}}$ by $-\frac{1}{13}x^{\frac{3}{4}}y^{\frac{4}{5}}z^{\frac{5}{6}}$.
9. Multiply $12x^3y^4z^5$ by $12xyz$.
10. Multiply $13xyz$ by $-13xyz$.
11. Find the product of $13x^2y \times \frac{1}{26}xy^2 \times 14xyz \times \frac{1}{7}xy^3$. *Ans.* $x^5y^7z$.
12. Find the product of $7xyz$, $-8x^2y^2z^2$ and $\frac{1}{28}x^3y^3z^3$. *Ans.* $-2x^6y^6z^6$.
13. Find the product of $7a^xb^yc^z$, $5ab^2c^3$ and $-\frac{1}{5}a^3b^2c$. *Ans.* $-7a^{x+4}b^{y+4}c^{z+4}$.
14. Find the product of $3a^xb^yz^c$ and $4a^mb^nz^p$. *Ans.* $12a^{x+m}b^{y+n}z^{c+p}$.

**60.** To multiply a polynomial by a monomial:

*Multiply each term of the multiplicand by the multiplier, according to* **59.**

EXAMPLES.

| | (1.) | (2.) | (3.) |
|---|---|---|---|
| Multiply | $x+y+z$ | $x-y-z$ | $x-y-z$ |
| by | $3$ | $4$ | $-5$ |
| *Ans.* | $3x+3y+3z$ | $4x-4y-4z$ | $-5x+5y+5z$ |

| | (4.) | (5.) |
|---|---|---|
| Multiply | $x^{\frac{1}{2}}+x^{\frac{1}{3}}+x^{\frac{1}{4}}$ | $x^2-2x^3+3xy-7m+12$ |
| by | $x^{\frac{1}{4}}$ | $xy$ |
| *Ans.* | $x^{\frac{3}{4}}+x^{\frac{7}{12}}+x^{\frac{1}{2}}$ | $x^3y-2x^4y+3x^2y^2-7mxy+12xy$ |

6. Multiply $x^2+xy+y^2$ by $x^2$. *Ans.* $x^4+x^3y+x^2y^2$.
7. Multiply $x^2+xy+y^2$ by $-xy$. *Ans.* $-x^3y-x^2y^2-xy^3$.
8. Multiply $x^2+xy+y^2$ by $y^2$. *Ans.* $x^2y^2+xy^3+y^4$
9. Multiply $x^3+x^2y+xy^2+y^3$ by $x$. *Ans.* $x^4+x^3y+x^2y^2+xy^3$.

10. Multiply $x^3 + x^2y + xy^2 + y^3$ by $-y$.

*Ans.* $-x^3y - x^2y^2 - xy^3 - y^4$.

11. Multiply $x^5 + x^4 + x^3 + x^2 + x + 1$ by $x$.

*Ans.* $x^6 + x^5 + x^4 + x^2 + x$.

12. Multiply $x^5 + x^4 + x^3 + x^2 + x + 1$ by $-1$.

*Ans.* $-x^5 - x^4 - x^3 - x^2 - x - 1$.

13. Multiply $x + y$ by $x$. *Ans.* $x^2 + xy$.

14. Multiply $x + y$ by $y$. *Ans.* $xy + y^2$.

**61.** To multiply one polynomial by another:

*Multiply every term of the multiplicand by each term of the multiplier, and add together the several products.*

EXAMPLES.

1. Multiply $x + y$
by $x + y$

$$\begin{array}{lrl}
\text{Multiply} & x + y & \\
\text{by} & x + y & \\
\hline
 & x^2 + xy \qquad & (\textit{Vide} \text{ above, ex. 13.}) \\
 & xy + y^2 & (\textit{Vide} \text{ above, ex. 14.}) \\
\hline
\text{Product} & x^2 + 2xy + y^2 & (\textit{Vide } \mathbf{34.})
\end{array}$$

2.
$$\begin{array}{ll}
\text{Multiply} & x^2 + xy + y^2 \\
\text{by} & x^2 - xy + y^2 \\
\hline
 & x^4 + x^3y + x^2y^2 \qquad\qquad\qquad (\textit{Vide} \text{ above, ex. 6.}) \\
 & \qquad - x^3y - x^2y^2 - xy^3 \qquad\quad (\textit{Vide} \text{ above, ex. 7.}) \\
 & \qquad\qquad\quad x^2y^2 + xy^3 + y^4 \quad (\textit{Vide} \text{ above, ex. 8.}) \\
\hline
\text{Product} & x^4 \qquad + x^2y^2 \qquad + y^4 \qquad (\textit{Vide } \mathbf{34.})
\end{array}$$

3.
$$\begin{array}{ll}
\text{Multiply} & x^3 + x^2y + xy^2 + y^3 \\
\text{by} & x - y \\
\hline
 & x^4 + x^3y + x^2y^2 + xy^3 \qquad\qquad (\textit{Vide} \text{ above, ex. 9.}) \\
 & \quad - x^3y - x^2y^2 - xy^3 - y^4 \quad (\textit{Vide} \text{ above, ex. 10.}) \\
\hline
\text{Product} & x^4 \qquad\qquad\qquad\qquad - y^4
\end{array}$$

4. Multiply $x^5 + x^4 + x^3 + x^2 + x + 1$

by $x - 1$

$$\begin{array}{ll}
x^6 + x^5 + x^4 + x^3 + x^2 + x & \text{(\textit{Vide} above, ex. 11.)} \\
\quad - x^5 - x^4 - x^3 - x^2 - x - 1 & \text{(\textit{Vide} above, ex. 12.)} \\
\hline
\text{Product } x^6 \qquad\qquad\qquad\qquad - 1 &
\end{array}$$

5. Multiply $x^2 + 5x + 7$

by $x^2 - 8x - 3$

$$\begin{array}{l}
x^4 + 5x^3 + 7x^2 \\
\quad - 8x^3 - 40x^2 - 56x \\
\qquad\quad - 3x^2 - 15x - 21 \\
\hline
x^4 - 3x^3 - 36x^2 - 71x - 21
\end{array}$$

Product $x^4 - 3x^3 - 36x^2 - 71x - 21$

6. Multiply $x^{\frac{1}{2}} + xy + y^{\frac{1}{2}}$

by $x^{\frac{1}{2}} - xy + y^{\frac{1}{2}}$

$$\begin{array}{l}
x + x^{\frac{3}{2}}y + x^{\frac{1}{2}}y^{\frac{1}{2}} \\
\quad - x^{\frac{3}{2}}y \qquad - x^2y^2 - xy^{\frac{3}{2}} \\
\qquad\quad + x^{\frac{1}{2}}y^{\frac{1}{2}} \qquad + xy^{\frac{3}{2}} + y \\
\hline
x \qquad + 2x^{\frac{1}{2}}y^{\frac{1}{2}} - x^2y^2 \qquad + y
\end{array}$$

Product $x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} - x^2y^2 + y$

7. Multiply $x + 5$ by $x + 6$. *Ans.* $x^2 + 11x + 30$.

8. Multiply $x + 5$ by $x - 6$. *Ans.* $x^2 - x - 30$.

9. Multiply $x - 8$ by $x - 9$. *Ans.* $x^2 - 17x + 72$.

10. Multiply $2x^2 + 3x - 1$ by $x - 5$.
*Ans.* $2x^3 - 7x^2 - 16x + 5$.

11. Find the product of $(x - 1)(x - 2)(x - 3)(x - 4)$.
*Ans.* $x^4 - 10x^3 + 35x^2 - 50x + 24$.

12. Find the product of $(x^2 - 2x + 5)(x + 1)$. *Ans.*

13. Find the product of $(x^2 + 2xy + y^2)(x^2 + 2xy + y^2)$.

14. Multiply $x^2 + 2ax + 7$ by $x^2 - ax + 5$.

15. Multiply $x^3 + 3x^2y + 3xy^2 + y^3$ by $x^2 + 2xy + y^2$.

16 Multiply $x^2 + 2xy + y^2$ by $x^2 - 2xy + y^2$.

17. Multiply $\quad x^3 + 3x^2y + 3xy^2 + y^3$

by $\quad x^3 - 3x^2y + 3xy^2 - y^3$

*Ans.* $\quad x^6 - 3x^4y^2 + 3x^2y^4 - y^6$

18 Multiply $\quad x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

by $\quad x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

*Ans.* $\quad x^8 - 4x^6y^2 + 6x^4y^4 - 4x^2y^6 + y^8$

19. Multiply $x^5 - x^4 + x^3 - x^2 + x - 1$ by $x + 1$. *Ans.* $x^6 - 1$.

20. Multiply $1 - x + x^2 - x^3$ by $1 + x + x^4 + x^5$.

21. Find the product of $(x - 5)\ (x - 6)\ (x - 7)\ (x + 8)$.
*Ans.* $x^4 - 10x^3 - 37x^2 + 646x - 1680$.

22. Multiply $x^{\frac{1}{2}} + x^{\frac{1}{3}}$ by $x^{\frac{1}{2}} - x^{\frac{1}{3}}$. *Ans.* $x - x^{\frac{2}{3}}$.

23. Multiply $x^{\frac{1}{2}} - x^{\frac{1}{3}} + x^{\frac{1}{4}}$ by $x^{\frac{1}{2}} + x^{\frac{1}{3}} - x^{\frac{1}{4}}$.
*Ans.* $x - x^{\frac{2}{3}} + 2x^{\frac{7}{12}} - x^{\frac{1}{2}}$.

24. Multiply $x^{\frac{1}{2}} + \frac{1}{3}x^3 - x$ by $x^{\frac{1}{2}} - \frac{1}{5}x^3 + x$.

25. Multiply $x^{\frac{1}{2}} + \frac{1}{3}x^{\frac{1}{3}} + \frac{1}{4}x^{\frac{1}{4}}$ by $x^{\frac{1}{2}} - 5x^{\frac{1}{3}} - 3x^{\frac{1}{4}}$.

26. Multiply $\frac{1}{5}x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{1}{3}}$ by $5x^{\frac{1}{2}} + 6x^{\frac{2}{3}}$.

27. Multiply $x^n + y^n$ by $x^n + y^n$. *Ans.* $x^{2n} + 2x^ny^n + y^{2n}$.

28. Find the product of $x^m - y^m$, $x^m + y^m$ and $x^n - y^n$.
*Ans.* $x^{2m+n} - x^ny^{2m} - x^{2m}y^n + y^{2m+n}$.

29. Find the product of $x^n + 2x^my^n + xy^p$ by $x^m - y^n$.

30. Find the value of $(x^2 + y^2)\ (x + y)\ (x - y)$ in a single polynomial.

31. Find the value of $(x^4 + y^4)\ (x^2 + y^2)\ (x + y)\ (x - y)$.

32. Find the value of $(x^2 - xy + y^2)\ (x^2 + xy + y^2)\ (x + y)$ $(x - y)$.

33. Find the value of $(x^2 + 5x + 7)\ (x^2 - 5x + 7)$.

34. Find the value of $(x^2 + 6x + 18)\ (x^2 - 6x + 18)$.

35. Find the value of $(x^2 + 4x + 8)\ (x^2 - 4x + 8)$.

36. Find the value of $(x^2 + 2ax + 2a^2)\ (x^2 - 2ax + 2a^2)$.

37. Find the value of $(x + 3)\ (x + 4)\ (x - 5)\ (x - 6)$.

38. Find the value of $(x - 5)\ (x - 6)\ (x - 4)$.

39. Multiply $x+5$ by $x+5$, $x-5$ by $x-5$, $x+7$ by $x+4$, and $x+7$ by $x-4$.

**62.** Since $(x+y)\ (x+y)=(x+y)^2=x^2+2xy+y^2$, it follows that

*The square of the sum of two quantities is equal to the square of the first + twice their product + the square of the last.*

EXAMPLES.

1. $(a+b)^2=a^2+2ab+b^2$.
2. $(2a+b)^2=4a^2+4ab+b^2$.
3. $(a+2b)^2=a^2+4ab+4b^2$.
4. $(3a+2b)^2=9a^2+12ab+4b^2$.
5. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})^2=x+2x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.
6. $(x^{\frac{3}{2}}+y^{\frac{3}{2}})^2=x^3+2x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3$.
7. $(2x^5+3x^4)^2=4x^{10}+12x^9+9x^8$.
8. $(x^2+5)^2=x^4+10x^2+25$.
9. Find the value of $(x^2+y)^2$, $(x^2+y^2)^2$, $(x^3+x)^2$, $(x^2+x)^2$, $(x^{\frac{1}{3}}+y^{\frac{1}{3}})^2$, $(x+4)^2$, $(x^2+x^3)^2$ and $(x^3+x^4)^2$, and the numerical value when $x=8$, $y=8$.

**63.** Since $(x-y)\ (x-y)=(x-y)^2=x^2-2xy+y^2$, it follows that

*The square of the difference of two quantities is equal to the square of the first — twice their product + the square of the last.*

EXAMPLES.

1. $(a-b)^2=a^2-2ab+b^2$.
2. $(2a-b)^2=4a^2-4ab+b^2$.
3. $(a-2b)^2=a^2-4ab+4b^2$
4. $(x-2)^2=x^2-4x+4$.
5. $(1-4x^2)^2=1-8x^2+16x^4$.
6. $(5-4)^2=25-40+16=1$.

7. $(1-1)^2 = 1 - 2 + 1 = 0.$
8. $(5x^{\frac{1}{2}} - x)^2 = 25x - 10x^{\frac{3}{2}} + x^2.$
9. Find the numerical value of $(x-3)^2$, $(x^2-5)^2$, $(1-x)^2$, $(3x^2-2x)^2$, $(4x^3-2x^3)^2$, $(5x^2-2y)^2$ and $(x-4)^2$ when $x=4$, $y=1$.

**64.** Since $(x+y)\ (x-y) = x^2 - y^2$, it follows that

*The product of the sum and difference of two quantities is equal to the difference of their squares.*

EXAMPLES.

1. $(a+b)\ (a-b) = a^2 - b^2.$
2. $(2a+b)\ (2a-b) = 4a^2 - b^2.$
3. $(x+4)\ (x-4) = x^2 - 16.$
4. $(x+5)\ (x-5) = x^2 - 25.$
5. $(5+2)\ (5-2) = (25-4) = 7 \times 3 = 21.$
6. $(4x^{\frac{3}{2}} + 4y^{\frac{3}{2}})\ (4x^{\frac{3}{2}} - 4y^{\frac{3}{2}}) = 16x^3 - 16y^3.$
7. $(3x^{\frac{1}{2}} + 2y^{\frac{1}{2}})\ (3x^{\frac{1}{2}} - 2y^{\frac{1}{2}}) = 9x - 4y.$
8. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})\ (x^{\frac{1}{2}} - y^{\frac{1}{2}}) = x - y.$
9. Find the numerical value of $(x^2+y^2)\ (x^2-y^2)$, $(x^3+y^3)\ (x^3-y^3)$ and $(x^4+y^4)\ (x^4-y^4)$ when $x=5$, $y=5$.

**65.** Since $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$, it follows that

*The square of a trinomial is equal to the sum of the squares of the three terms united to twice the product of the terms taken two and two.* (For the signs observe **59.**)

EXAMPLES.

1. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$
2. $(x+y-z)^2 = x^2 + y^2 + z^2 + 2xy - 2xz - 2yz.$
3. $(x-y-z)^2 = x^2 + y^2 + z^2 - 2xy - 2xz + 2yz.$
4. $(x^2+x+1)^2 = x^4 + x^2 + 1 + 2x^3 + 2x^2 + 2x = x^4 + 2x^3 + 3x^2 + 2x + 1.$

5. $(5x^2 - 4x + 3)^2 = 25x^4 - 40x^3 + 46x^2 - 24x + 9$.

6. $(3x^3 + 4x^2 + 5x)^2 = 9x^6 + 24x^5 + 46x^4 + 40x^3 + 25x^2$.

7. Find the value of $(9x^2 - 6x - 2)^2$, $(x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}})^2$, $(x + 2y + 3z)^2$ and the numerical value when $x = 4$, $y = 9$, $z = 16$.

8. Find the values of $(3x + 4y)^2$, $(3x - 4y)^2$, $(3x + 4y)\ (3x - 4y)$, and $(1 + 3x - 4y)^2$, and the numerical values when $x = 2$, $y = 1$.

**66.** 1. Multiply $(x - a)$, $(x - b)$ and $(x - c)$ together. (*Vide* Def. **12.**)

*Operation.*

$$\begin{array}{l|l|l}
x - a & & \\
x - b & & \\
\hline
x^2 - a & x + ab & \\
\quad\ - b & & \\
x - c & & \\
\hline
x^3 - a & x^2 + ac & x - abc \\
\quad\ - b & \quad\ + bc & \\
\quad\ - c & \quad\ + ab &
\end{array}$$

*Ans.* (as above)

2. What would be the answer to the last example when $a = 1$, $b = 2$, $c = 3$? *Ans.* $x^3 - 6x^2 + 11x - 6$.

3. What would be the value when $a = 2$, $b = 3$, $c = 4$? *Ans.* $x^3 - 9x^2 + 26x - 24$.

4. Find the product of $(x - a)\ (x - b)\ (x - c)\ (x - d)$, and what will it become when $a = 1$, $b = 2$, $c = 3$, $d = 4$; also when $a = 3$, $b = 4$, $c = 5$, $d = 6$. (*Vide* **61**, ex. 11.)

*Ans.* to last $x^4 - 18x^3 + 119x^2 - 342x + 360$.

# DIVISION.

**67.** Division in Algebra is the operation of *finding a quotient which multiplied into a given divisor will produce a given dividend.*

**68.** To divide one monomial by another:

By **56** we have $7a^2b \times 5ax = 35a^3bx$ $\therefore$ $35a^3bx \div 7a^2b = 5ax$.

By **57** we have $4a^3b \times -2a^2b = -8a^5b^2$ $\therefore$ $-8a^5b^2 \div 4a^3b = -2a^2b$.

Also, (example 10) $3x^{\frac{1}{7}}y^{-5} \times 5x^{\frac{1}{7}}y^4 = 15x^{\frac{2}{7}}y^{-1}$ $\therefore$ $15x^{\frac{2}{7}}y^{-1} \div 5x^{\frac{1}{7}}y^4 = 3x^{\frac{1}{7}}y^{-5}$.

Hence,

(1.) *Divide the coëfficient of the dividend by that of the divisor, and annex all the letters, giving to each an exponent found by subtracting the exponent of a letter in the divisor from the exponent of the same letter in the dividend.*

(2.) If the dividend and divisor have *like* signs, the quotient is *plus.*

(3.) If the dividend and divisor have *unlike* signs, the quotient is *minus.*

(4.) If the divisor contains letters not found in the dividend, these letters may be inserted in the dividend by giving to each the exponent 0. (*Vide* Def. **13**, 9.)

Thus, $\dfrac{4x^2}{abc} = \dfrac{4a^0b^0c^0x^2}{abc}$.

EXAMPLES.

1. $a \div a = \frac{a}{a} = a^0 = 1$. (*Vide* Def. **13**, 9.)

2. $a^2 \div a = \frac{a^2}{a} = a$.

3 $a^3 \div a = \frac{a^3}{a} = a^2$.

4. $a \div a^2 = \frac{a}{a^2} = \frac{1}{a} = a^{-1}$.

5. $ax \div axy = \frac{axy^0}{axy} = \frac{y^0}{y} = y^{-1}$.

6. $10a^2x^3 \div 5a^{-1}x^4 = 2a^3x^{-1}$.

7. $-100x^2y^3z \div -50x^3y^4z^2 = 2x^{-1}y^{-1}z^{-1}$.

8. $7xy \div 5m = \frac{7xym^0}{5m} = \frac{7}{5}xym^{-1}$.

9. $21a^3b^5 \div 7a^2b^4$, $57a^4b^2 \div 19ab$, $15a^2bx \div 3a$, $35a^2b^2x^2 \div 7a$.

10. $27a^3b^3c^3 \div 9ac$, $-33x^5y^6 \div 11xy$, $-42x^3y^4 \div 21x^2y^3$, $45x^4y^3 \div 15xy$.

11. $7x^5y^4 \div -35xy$, $-20x^3y \div 4xy^2$, $3a^2 \div 2b$, $7a^2mn \div 3a^3$.

12. $-26x^5y \div -13xy$, $16xy \div 8xy$, $-30xy^3 \div 15xy$, $72x^8y^8 \div 36x^4y^4$.

13. $x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}} \div x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} = x^{\frac{1}{6}}y^{\frac{1}{6}}z^{\frac{1}{6}}$.

14. $10x^{\frac{1}{5}}y^{\frac{2}{3}}z^{\frac{1}{4}} \div 5x^{\frac{1}{6}}y^{\frac{1}{3}}z^{\frac{1}{3}} = 2x^{\frac{1}{30}}y^{\frac{1}{3}}z^{-\frac{1}{12}}$

15. $-6\frac{3}{4}xy^5 \div -2\frac{1}{4}xy^4 = 3y$.

16. $-7\frac{3}{4}x^5y \div -2\frac{7}{12}xy = 3x^4$.

17. $729xy \div -27x^{\frac{1}{3}}y^{\frac{1}{3}}$, $-225x^{\frac{1}{3}}y^{\frac{1}{3}} \div -15x^{-\frac{2}{3}}y^{\frac{1}{6}}$, $-12\frac{1}{2}x^{\frac{2}{3}}y^{\frac{1}{4}} \div 6\frac{1}{4}x^{\frac{1}{3}}y^{\frac{1}{8}}$.

18. $441x^2y^4 \div -21xy^3$, $361x^7y^8 \div -19x^6y^7$, $13xy \div 15xyz$.

**69.** To divide a polynomial by a monomial:

*Divide each term of the dividend by the divisor, according to* **68.**

EXAMPLES.

1. Divide $x^2 + 2xy + y^2$ by $x$. *Ans.* $x + 2y + y^2x^{-1}$.
2. Divide $3ax^2 + 6abx - 9x^3$ by $3x$. *Ans.* $ax + 2ab - 3x^2$.
3. Divide $xy + xz + yz$ by $xyz$. *Ans.* $z^{-1} + y^{-1} + x^{-1}$.
4. Divide $xyz + xyw + xzw + yzw$ by $xyzw$.

   *Ans.* $w^{-1} + z^{-1} + y^{-1} + x^{-1}$.
5. Divide $-6x^5 + 30x^3 - 9ax^2 + 6mx - 3nx$ by $-3x$.
6. Divide $12a^x + 24a^y - 36a^z$ by $12a$. *Ans.* $a^{x-1} + 2a^{y-1} - 3a^{z-1}$.
7. Divide $4x^2(x+y)^2 - 8x^3(x+y)^3 + 12x^4(x+y)^5$ by $4x^2(x+y)^2$.

   *Ans.* $1 - 2x(x+y) + 3x^2(x+y)^3$.

8. Divide $7x^{\frac{1}{3}}(x+y+z) + 4x^{\frac{1}{2}}(x+y+z)$ by $3x^{\frac{1}{3}}(x+y+z)$.

*Ans.* $2\frac{1}{3} + \frac{4}{3}x^{\frac{1}{6}}$.

**70.** To divide one polynomial by another:

(1.) *Arrange the polynomials with reference to the ascending or descending powers of the same letter.*

(2.) *Divide the first term of the dividend by the first term of the divisor for the first term of the quotient. Multiply the whole divisor by this term of the quotient, and subtract the product from the dividend for the first remainder.*

(3.) *Upon the first remainder repeat the very same operation as upon the given dividend, for the second remainder, and so on till there is no remainder; or, if the division be not exact, till the first term of the remainder divided by the first term of the divisor would produce a negative exponent in the quotient. In this case place the remainder over the divisor at the right of the quotient, prefixing the proper sign.*

(4.) It often happens that the division may be carried on indefinitely, giving rise to an *infinite series,* in which case a few of the leading terms of the quotient will generally be sufficient to determine the rest.

EXAMPLES.

1. Divide $x^2 + 2xy + y^2$ by $x + y$.

*Solution.*

$$\begin{array}{l|l}
\text{Dividend } x^2 + 2xy + y^2 & x + y = \text{Divisor.} \quad (\textit{Vide } \text{Def. } \mathbf{11.}) \\
\cline{2-2}
\underline{x^2 + xy} & x + y = \text{Quotient.} \\
\qquad xy + y^2 & \\
\qquad \underline{xy + y^2} &
\end{array}$$

*Explanation.*

The first term, $x^2$, of the dividend is divided by $x$, the first term of the divisor, and we obtain $x$, the first term of the quo-

tient. Next, multiply the divisor $x+y$ by $x$, and we obtain the product $x^2+xy$, which, subtracted from the dividend, gives $xy+y^2$ for the remainder. Divide its first term $xy$ by $x$, and we obtain $y$ for the second term of the quotient. Multiply $x+y$ by $y$, and we have $xy+y^2$, which being taken from the remainder leaves nothing, the quotient being exact. (*Vide* **62** & **61.**)

2. Divide $x^2-2xy+y^2$ by $x-y$. *Ans.* $x-y$.
3. Divide $x^3+3x^2y+3xy^2+y^3$ by $x^2+2xy+y^2$. *Ans.* $x+y$.
4. Divide $x^3-3x^2y+3xy^2-y^3$ by $x-y$. *Ans.* $x^2-2xy+y^2$.
5. Divide $x^4+4x^3y+6x^2y^2+4xy^3+y^4$ by $x^2+2xy+y^2$.

   *Ans.* $x^2+2xy+y^2$.
6. Divide $x^4-3x^3-36x^2-71x-21$ by $x^2-8x-3$.

*Operation.*

$$\begin{array}{lrl}
\text{Dividend.} & x^4-3x^3-36x^2-71x-21 & \big|\ x^2-8x-3 \quad \text{Divisor.} \\
& x^4-8x^3-3x^2 \qquad\qquad\quad & \ \ x^2+5x+7 \quad \text{Quotient.} \\
\hline
& 5x^3-33x^2-71x-21 & \\
& 5x^3-40x^2-15x \quad\ \ & \\
\hline
& 7x^2-56x-21 & \\
& 7x^2-56x-21 & \\
\hline
\end{array}$$

7. Divide $x^3-x^2+3x+5$ by $x+1$. *Ans.* $x^2-2x+5$.
8. Divide $4x^6-25x^2+20x-4$ by $2x^3-5x+2$.

   *Ans.* $2x^3+5x-2$.
9. Divide $2x^3-7x^2-16x+5$ by $x-5$. *Ans.* $2x^2+3x-1$.
10. Divide $x^6-3x^4y^2+3x^2y^4-y^6$ by $x^3-3x^2y+3xy^2-y^3$.

*Operation.*

$$\begin{array}{l|l} x^6 - 3x^4y^2 + 3x^2y^4 - y^6 & x^3 - 3x^2y + 3xy^2 - y^3 \\ \cline{2-2} x^6 - 3x^5y + 3x^4y^2 - x^3y^3 & x^3 + 3x^2y + 3xy^2 + y^3 \\ \cline{1-1} \quad 3x^5y - 6x^4y^2 + x^3y^3 + 3x^2y^4 - y^6 & \\ \quad 3x^5y - 9x^4y^2 + 9x^3y^3 - 3x^2y^4 & \\ \cline{1-1} \qquad 3x^4y^2 - 8x^3y^3 + 6x^2y^4 - y^6 & \\ \qquad 3x^4y^2 - 9x^3y^3 + 9x^2y^4 - 3xy^5 & \\ \cline{1-1} \qquad\quad x^3y^3 - 3x^2y^4 + 3xy^5 - y^6 & \\ \qquad\quad x^3y^3 - 3x^2y^4 + 3xy^5 - y^6 & \\ \cline{1-1} \end{array}$$

11. Divide $x^4 - 2x^2y^2 + y^4$ by $x^2 - 2xy + y^2$.

*Ans.* $x^2 + 2xy + y^2$.

12. Divide $x^8 - 4x^6y^2 + 6x^4y^4 - 4x^2y^6 + y^8$ by $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$. *Ans.* $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

13. Divide $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$. *Ans.* $x^2 - xy + y^2$.

14. Divide $x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} - x^2y^2 + y$ by $x^{\frac{1}{2}} - xy + y^{\frac{1}{2}}$.

*Operation.*

$$\begin{array}{l|l} -x^2y^2 + x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y & -xy + x^{\frac{1}{2}} + y^{\frac{1}{2}} \\ \cline{2-2} -x^2y^2 + x^{\frac{3}{2}}y + xy^{\frac{3}{2}} & \quad xy + x^{\frac{1}{2}} + y^{\frac{1}{2}} \\ \cline{1-1} \quad -x^{\frac{3}{2}}y - xy^{\frac{3}{2}} + x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y & \\ \quad -x^{\frac{3}{2}}y \qquad\quad + x + x^{\frac{1}{2}}y^{\frac{1}{2}} & \\ \cline{1-1} \qquad -xy^{\frac{3}{2}} \qquad + x^{\frac{1}{2}}y^{\frac{1}{2}} + y & \\ \qquad -xy^{\frac{3}{2}} \qquad + x^{\frac{1}{2}}y^{\frac{1}{2}} + y & \\ \cline{1-1} \end{array}$$

15. Divide $x - x^{\frac{2}{3}}$ by $x^{\frac{1}{2}} + x^{\frac{1}{3}}$. *Ans.* $x^{\frac{1}{2}} - x^{\frac{1}{3}}$.

16. Divide $x - x^{\frac{2}{3}} + 2x^{\frac{7}{12}} - x^{\frac{1}{2}}$ by $x^{\frac{1}{2}} + x^{\frac{1}{3}} - x^{\frac{1}{4}}$.

*Ans.* $x^{\frac{1}{2}} - x^{\frac{1}{3}} + x^{\frac{1}{4}}$.

17. Divide $x - \frac{14}{3}x^{\frac{5}{6}} - \frac{11}{4}x^{\frac{3}{4}} - \frac{5}{3}x^{\frac{2}{3}} - \frac{9}{4}x^{\frac{7}{12}} - \frac{3}{4}x^{\frac{1}{2}}$ by $x^{\frac{1}{2}} - 5x^{\frac{1}{3}} - 3x^{\frac{1}{4}}$.

*Ans.* $x^{\frac{1}{2}} + \frac{1}{3}x^{\frac{1}{3}} + \frac{1}{4}x^{\frac{1}{4}}$.

18. Divide $x - y^{\frac{2}{3}} + 2y^{\frac{1}{3}}z^{\frac{1}{4}} - z^{\frac{1}{2}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{3}} + z^{\frac{1}{4}}$.

*Ans.* $x^{\frac{1}{2}} + y^{\frac{1}{3}} - z^{\frac{1}{4}}$.

19. Divide $x^2 + 5x + 7$ by $x - 3$.

*Operation.*

$$\begin{array}{l|l} x^2 + 5x + 7 & x - 3 \\ \cline{2-2} x^2 - 3x & x + 8 + \frac{31}{x-3} \\ \cline{1-1} \quad 8x + 7 & \\ \quad 8x - 24 & \\ \cline{1-1} \qquad 31 & \end{array}$$

20. Divide $x^2 - 6x - 41$ by $x + 3$. *Ans.* $x - 9 - \frac{14}{x+3}$.

21. Divide $x^3 + 4x^2 + 5x + 6$ by $x^2 + 3x - 1$.

*Ans.* $x + 1 + \frac{3x+7}{x^2+3x-1}$

22. Divide $1 + y^2$ by $1 + x$.

*Operation.*

$$\begin{array}{l|l} 1 + y^2 & 1 + x \\ \cline{2-2} 1 + x & 1 - x + x^2 - x^3 + \text{ \&c.} \\ \cline{1-1} \; - x + y^2 & \\ \; - x - x^2 & \\ \cline{1-1} \qquad x^2 + y^2 & \\ \qquad x^2 + x^3 & \\ \cline{1-1} \qquad\quad - x^3 + y^2 & \\ \qquad\quad - x^3 - x^4 & \\ \cline{1-1} \end{array}$$

23. Divide $1 + 2x + 3x^2$ by $1 - 4x$.

*Ans.* $1 + 6x + 27x^2 + 108x^3 + 432x^4 +$ &c.

24. Divide $1 - x$ by $1 + x$. *Ans.* $1 - 2x + 2x^2 - 2x^3$ &c.

25. Divide $1 - x$ by $1 + x + x^2$.

*Ans.* $1 - 2x + x^2 + x^3 - 2x^4 + x^5 + x^6 - 2x^7$ &c.

26. Divide $1 + 2x$ by $1 - x - x^2$.

*Ans.* $1 + 3x + 4x^2 + 7x^3 +$ &c.

27. Divide $x^{4n}+x^{2n}y^{2n}+y^{4n}$ by $x^{2n}+x^ny^n+y^{2n}$.

*Operation.*

$$
\begin{array}{l|l}
x^{4n}+x^{2n}y^{2n}+y^{4n} & x^{2n}+x^ny^n+y^{2n} \\ \cline{2-2}
x^{4n}+x^{3n}y^n+x^{2n}y^{2n} & x^{2n}-x^ny^n+y^{2n} \\ \cline{1-1}
\quad -x^{3n}y^n+y^{4n} & \\
\quad -x^{3n}y^n-x^{2n}y^{2n}-x^ny^{3n} & \\ \cline{1-1}
\qquad\qquad x^{2n}y^{2n}+x^ny^{3n}+y^{4n} & \\
\qquad\qquad x^{2n}y^{2n}+x^ny^{3n}+y^{4n} & \\ \cline{1-1}
\end{array}
$$

28. Divide $x^{m+n}+2x^{2m}y^n+x^{m+1}y^p-x^ny^n-2x^my^{2n}-xy^{p+n}$ by $x^m-y^n$. *Ans.* $x^n+2x^my^n+xy^p$.

**71.** 1. Divide $x^3-y^3$ by $x-y$. *Ans.* $x^2+xy+y^2$.

2. Divide $x^2-y^2$ by $x-y$. *Ans.* $x+y$.

3. Divide $x^3+y^3$ by $x+y$. *Ans.* $x^2-xy+y^2$.

4. Divide $x^2-y^2$ by $x+y$. *Ans.* $x-y$.

5. Divide $x^2+y^2$ by $x-y$. *Ans.* $x+y+\frac{2y^2}{x-y}$.

6. Divide $x^3+y^3$ by $x-y$. *Ans.* $x^2+xy+y^2+\frac{2y^3}{x-y}$.

7. Divide $x^3-y^3$ by $x+y$. *Ans.* $x^2-xy+y^2-\frac{2y^3}{x+y}$.

8. Divide $x^2+y^2$ by $x+y$. *Ans.* $x-y+\frac{2y^2}{x+y}$.

(1.) For the present we may infer from 1 and 2 that the *difference of two quantities will divide the difference of the like powers of those quantities, without a remainder.* (*Vide* **78.**)

(2.) From 3 and 4, the *sum of two quantities will divide the sum of the like odd or difference of the like even powers of those quantities, without a remainder.* (*Vide* **79.**)

(3.) From 5 and 6, the *difference* of two quantities will not divide the *sum* of the like powers of those quantities, without a remainder.

(4.) From 7 and 8 the *sum* of two quantities will not divide the *difference* of the like *odd* or the *sum* of the like *even* powers of those quantities, without a remainder.

**72.** All the above cases may be solved mentally by observing the following directions:

(1.) The exponent of the leading letter $x$ *decreases* by *one* regularly, and this letter disappears in the last term.

(2.) The exponent of $y$ *increases* by *one* regularly from the second term.

(3.) If the divisor contain the *negative* sign, then the terms of the quotient will all be *positive*.

(4.) If *both* terms of the divisor are *positive*, then the *odd* terms of the quotient are *positive*, the *even* terms *negative*.

(5.) In the cases which have a remainder, this remainder will always be twice the given power of $y$, *retaining its sign*.

EXAMPLES.

1. Divide $x^5 - y^5$ by $x - y$. *Ans.* $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.
2. Divide $x^4 - y^4$ by $x - y$. *Ans.* $x^3 + x^2y + xy^2 + y^3$.
3. Divide $x^5 + y^5$ by $x + y$. *Ans.* $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.
4. Divide $x^4 - y^4$ by $x + y$. *Ans.* $x^3 - x^2y + xy^2 - y^3$.
5. Divide $x^4 + y^4$ by $x - y$. *Ans.* $x^3 + x^2y + xy^2 + y^3 + \frac{2y^4}{x-y}$.
6. Divide $x^5 + y^5$ by $x - y$.
   *Ans.* $x^4 + x^3y + x^2y^2 + xy^3 + y^4 + \frac{2y^5}{x-y}$.
7. Divide $x^5 - y^5$ by $x + y$.
   *Ans.* $x^4 - x^3y + x^2y^2 - xy^3 + y^4 - \frac{2y^5}{x+y}$.
8. Divide $x^4 + y^4$ by $x + y$. *Ans.* $x^3 - x^2y + xy^2 - y^3 + \frac{2y^4}{x+y}$
9. Find the value of $\frac{x-y}{x-y}$, $\frac{x^6-y^6}{x-y}$, $\frac{x^7-y^7}{x-y}$, $\frac{x^8-y^8}{x-y}$, $\frac{x+y}{x+y}$, $\frac{x^7+y^7}{x+y}$ and $\frac{x^9+y^9}{x+y}$.

10. Find the value of $\frac{x^8 - y^8}{x + y}$, $\frac{x^6 - y^6}{x + y}$, $\frac{x^7 - y^7}{x + y}$, $\frac{x^9 - y^9}{x + y}$, $\frac{x^7 + y^7}{x - y}$ and $\frac{x^{10} + y^{10}}{x + y}$.

11. Divide $x^m - y^m$ by $x - y$. *Ans.* $x^{m-1} + x^{m-2}y + x^{m-3}y^2$ &c. $+ xy^{m-2} + y^{m-1}$.

12. Find the value of $\frac{x^2 - 1}{x - 1}$, $\frac{x^3 - 1}{x - 1}$, $\frac{x^5 + 32}{x + 2}$, $\frac{x^3 - 27}{x - 3}$, $\frac{x^5 + 243}{x + 3}$ and $\frac{81x^4 - 16y^4}{3x - 2y}$.

To find the answers of examples 12, make in 11, $m = 2$, $y = 1$, $m = 3$, $y = 1$; $m = 5$, $y = 2$; $m = 3$, $y = 3$; $m = 5$, $y = 3$; $m = 4$, and place in the last $3x$ for $x$ and $2y$ for $y$, using as many terms of 11 as is indicated by the value of $m$.

*Ans.* to the last $27x^3 + 18x^2y + 12xy^2 + 8y^3$.

13. Divide $x^{mn} - y^{mn}$ by $x^m - y^n$.

*Ans.* $x^{mn-n} + x^{mn-2n}y^n + x^{mn-3n}y^{2n} +$ &c. $+ y^{mn-n}$.

14. Find the value of $\frac{x^4 - y^4}{x^2 - y^2}$, $\frac{x^6 - y^6}{x^3 - y^3}$, $\frac{x^8 - y^8}{x^4 - y^4}$ and $\frac{x^{12} - y^{12}}{x^3 - y^3}$.

15. If in ex. 1 $x = y$, what does the answer become?

*Ans.* $5x^4$ or $5y^4$.

16. If in ex. 2 $x = y$, what does the answer become?

*Ans.* $4x^3$ or $4y^3$.

17. If in ex. 11 $x = y$, what does the answer become?

*Ans.* $mx^{m-1}$ or $my^{m-1}$.

18. If in ex. 16 and 17 $x = 5$ and $m = 4$, what will they become? *Ans.* 500.

Factoring.

Chapter III

# CHAPTER III.

## FACTORING—GREATEST COMMON DIVISOR—LEAST COMMON MULTIPLE.

### FACTORING.

**73.** FACTORING is the operation of *resolving a quantity into factors.*

1. A *composite quantity* is one which may be resolved into *factors.*

2. A *prime* factor cannot be resolved into other factors.

3. All the cases of factoring merely reverse the operations of multiplication.

**74.** To separate a monomial into its prime factors:

*Separate the coëfficient into its prime factors, and annex the literal part, also resolved.*

EXAMPLES.

1. Find the factors of $12a^2bx$. *Ans.* 2. 2. 3. $a$. $a$. $b$. $x$.
2. Find the factors of $16a^3$, $169x^3y$, $112x^4y^2$ and $133x^2y^2$.

**75.** To factor a polynomial when multiplied into a monomial:

*Divide the polynomial by the monomial common to all the terms. The divisor and quotient are the factors.* (*Vide* **60** and **12.**)

EXAMPLES.

1. Find the factors of $x^2 + xy$. *Ans.* $(x + y)\,x$.
2. Find the factors of $xy + y^2$. *Ans.* $(x + y)\,y$.

3. Find the factors of $ax + bx$. *Ans.* $(a + b)\,x$.
4. Factor $ax + bx + cx$, $a^2x^2 + b^2x^2$, $5x + 2bx$ and $3x^2y + 3xy^2$.
5. Factor $x^3y - x^2y^2 + xy^3$, $x^3y^2 + x^2y^3$ and $5ax - 2bx + x$.
6. Factor $x + ax - 2bx$, $5bx + x - ax$ and $3ax + 6bx - 12cx$.
7. Find the factors of $x^{m-1}y - y^m$. *Ans.* $(x^{m-1} - y^{m-1})\,y$.
8. Find the factors of $x + y + (l + m)\,(x + y)$.
   *Ans.* $(1 + l + m)\,(x + y)$.
9. $(x + a)\,(x + b) = x^2 + (a + b)\,x + ab$.
10. $(x - a)\,(x - b) = x^2 - (a + b)\,x + ab$.
11. $(x - a)\,(x + b) = x^2 - (a - b)\,x - ab$.
12. $(x + a)\,(x - b) = x^2 + (a - b)\,x - ab$.

**76.** To factor expressions of the form $x^2 + 2xy + y^2$:

*Take the square root of the extreme terms, and place the sign between the roots, that is, before the middle term.* The result is one of the equal factors. (*Vide* **62** and **63.**)

EXAMPLES.

1. Find the factors of $x^2 + 2xy + y^2$. *Ans.* $(x + y)\,(x + y)$.
2. Find the factors of $x^4 + 2x^2y^2 + y^4$. *Ans.* $(x^2 + y^2)\,(x^2 + y^2)$.
3. Factor $64x^4 - 96x^3 + 36x^2$, $1 + 2x + x^2$, $1 - 2x + x^2$ and $1 - 2x^{\frac{1}{3}} + x^{\frac{2}{3}}$.
4. Factor $1 + \frac{2a^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \frac{a}{x}$, $\frac{a^2}{x^2} - 2 + \frac{x^2}{a^2}$, $1 + x^{\frac{1}{3}} + \frac{1}{4}x^{\frac{2}{3}}$ and $1 + \frac{1}{x} + \frac{1}{4x^2}$.
5. Factor $x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y$, $x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$, $25x - 20x^{\frac{3}{2}} + 4x^2$ and $x^{\frac{2}{3}} - 2x + x^{\frac{4}{3}}$.
6. Factor $1 + \frac{1}{x^{\frac{1}{2}}} + \frac{1}{4x}$, $x^2 - 5x + \frac{25}{4}$, $x^2 + 14x + 49$ and $x^2 - \frac{3}{4}x + \frac{9}{64}$.

**77.** To factor expressions of the form $x^2 - y^2$:

*Indicate the product of the sum and the difference of the square roots of the quantities.* (*Vide* **64.**)

EXAMPLES.

1. Factor $x^2 - y^2$. *Ans.* $(x+y)(x-y)$.
2. Factor $9x^2 - 4y^2$. *Ans.* $(3x+2y)(3x-2y)$
3. Factor $1-4x^2$, $1-9x^2$, $4-16y^2$, $9x^2-4y^4$ and $1-\frac{x^2}{4}$.
4. Factor $x^4 - x^2$. *Ans.* $x^2(x^2-1) = x^2(x+1)(x-1)$.
5. Factor $x^4 - y^4$.
*Ans.* $(x^2+y^2)(x^2-y^2) = (x^2+y^2)(x+y)(x-y)$.
6. Factor $x^8-y^8$, $x^{16}-y^{16}$, $x^{32}-y^{32}$, $16x^4-16y^4$ and $16x^4-81y^4$.
7. Factor $x-y$, $x^{\frac{2}{3}}-y^{\frac{2}{3}}$, $4x^2-y$, $x^{\frac{1}{2}}-y^{\frac{1}{2}}$ and $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.
8. Factor $(x+p)^2-q^2$. *Ans.* $(x+p+q)(x+p-q)$.

**78.** To factor expressions of the form $x^m - y^m$, $m$ being any positive whole number:

$$\text{Since } \frac{x^m-y^m}{x-y} = x^{m-1} + y\frac{x^{m-1}-y^{m-1}}{x-y}, \text{ (vide 75, ex. 7)},$$

it follows that when $x^{m-1}-y^{m-1}$ is exactly divisible by $x-y$ then $x^m-y^m$ must also be divisible by $x-y$; but $x^2-y^2$ is exactly divisible by $x-y$, the quotient being $x+y$ ∴ the difference of two quantities, &c. (*Vide* **71**, (1).) Hence,

*To factor $x^m-y^m$, divide by $x-y$ and indicate the product of the divisor and quotient.*

**79.** To factor expressions of the form $x^m+y^m$, $m$ being any positive *odd* number:

$$\text{Since } \frac{x^m+y^m}{x+y} = x^{m-1} - x^{m-2}y + y^2\frac{x^{m-2}+y^{m-2}}{x+y},$$

it follows that when $x^{m-2}+y^{m-2}$ is exactly divisible by $x+y$, then $x^m+y^m$ is divisible by $x+y$; but $x^3+y^3$ is divisible by $x+y$, the quotient being $x^2-xy+x^2$ ∴ the sum of any two quantities will always divide the sum of the like *odd* powers of the same quantities. (*Vide* **71**, (2).)

In the same way we may show that the sum of two quan-

tities will always divide the difference of the like even powers of the same quantities. Hence,

To factor $x^m - y^m$ when $m$ is *odd*, divide by $x + y$ and indicate the product of the divisor and quotient.

To factor $x^m - y^m$ when $m$ is *even*, divide by $x + y$ or by $x - y$, &c.

EXAMPLES.

1. $x^3 + y^3 = (x + y)\ (x^2 - xy + y^2)$.
2. $x^3 - y^3 = (x - y)\ (x^2 + xy + y^2)$.
3. $x^6 - y^6 = (x^3 + y^3)\ (x^3 - y^3) = (x + y)\ (x^2 - xy + y^2)\ (x - y)$ $(x^2 + xy + y^2)$.
4. $x^4 - y^4 = (x^2 + y^2)\ (x + y)\ (x - y)$, or $(x - y)\ (x^3 + x^2y + xy^2 + y^3)$, or $(x + y)\ (x^3 - x^2y + xy^2 - y^3)$.
5. Factor $x^6 - y^6$, $x^5 + y^5$, $x^5 - y^5$, $x^{10} - y^{10}$ and $x^{12} - y^{12}$.
6. Is $x^2 + y^2$ a composite quantity?
7. In how many ways may $x^8 - y^8$ be factored?

*Ans.* $x^8 - y^8 = (x^4 + y^4)\ (x^2 + y^2)\ (x + y)\ (x - y)$.

$x^8 - y^8 = (x - y)\ (x^7 + x^6y + x^5y^2$, &c.

$x^8 - y^8 = (x + y)\ (x^7 - x^6y + x^5y^2 -$, &c.

$x^8 - y^8 = (x^2 + y^2)\ (x^6 - x^4y^2 + x^2y^4 - y^6)$.

$x^8 - y^8 = (x^2 - y^2)\ (x^6 + x^4y^2 + x^2y^4 + y^6) = (x + y)$ $(x - y)\ (x^6 + x^4y^2 + x^2y^4 + y^6)$.

8. Find all the factors of $x^{12} - y^{12}$ and $x^{16} - y^{16}$.

**80.** To factor an expression of the form $x^2 + (a + b)\,x + ab$, $a$ and $b$ being positive or negative whole numbers, (*vide* **75**, ex. 9, 10, 11 and 12):

*Find by inspection $a$ and $b$, and indicate the factors thus:* $(x + a)\ (x + b)$.

EXAMPLES.

1. $x^2 + 9x + 20 = (x + 5)\ (x + 4)$.
2. $x^2 - 9x + 20 = (x - 5)\ (x - 4)$.

3. $x^2 + x - 20 = (x + 5)(x - 4)$.
4. $x^2 - x - 20 = (x - 5)(x + 4)$.

If the signs of the trinomial are all positive the factors are positive.

If the middle sign *only* is negative both *numbers* are negative.

If the last sign *only* is negative the *largest* number is *positive*.

If the middle and last signs are negative the *largest* number is *negative*.

5. Factor $x^2 - 13x + 42$, $x^2 + 13x + 42$, $x^2 - x - 42$ and $x^2 + x - 42$.
6. Factor $x^2 + x - 12$, $x^2 - x - 12$, $x^2 - x - 30$ and $x^2 + 2x - 360$.
7. Factor $x^2 + 11x + 24$, $x^4 - 3x^2 - 4$, $x^4 + 10x^2 + 9$ and $x^4 - 3x^2 + 2$.
8. $x^4 - 10x^2 + 9 = (x^2 - 9)(x^2 - 1) = (x + 3)(x - 3)(x + 1)(x - 1)$.
9. $x^4 - 17x^2 + 16 = (x + 4)(x - 4)(x + 1)(x - 1)$.
10. Factor $x^4 - 37x^2 + 36$, $x^6 - 26x^3 + 25$ and $x^4 - 40x^2 + 144$.
11. Factor $4x^2 - 4x - 80$. *Ans.* $4(x - 5)(x + 4)$.
12. Factor $7x^2 - 7x - 84$, $5x^2 - 5x - 60$ and $x^3 - 13x^2 + 42x$.

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## GREATEST COMMON DIVISOR.

**81.** The Greatest Common Divisor of two or more quantities is the greatest quantity that will divide each of them without a remainder. Thus, $xy$ is the greatest common divisor of $x^2y$ and $xy^2$.

**82.** To find the greatest common divisor of two or more quantities:

*Find the product of the prime factors common to all the quantities.*

EXAMPLES.

1. Find the greatest common divisor of $a^2b$, $ab^2$ and $abc$.
*Ans.* $ab$.

2. Find the greatest common divisor of $3x^2y + 3xy^2$ and $6x^2y^2 + 6x^3y$.
*Ans.* $3xy$.

3. Find the greatest common divisor of $x^2 + x - 6$ and $x^2 + 8x + 15$.
*Ans.* $x + 3$.

4. Find the greatest common divisor of $x^3 - y^3$ and $x^2 - y^2$.
*Ans.* $x + y$.

5. Find the greatest common divisor of $x^2 - 16$ and $x^2 - x - 20$; $x^2 + 15x + 56$ and $x^2 + 5x - 14$.

6. Find the greatest common divisor of $x^2 + 2x - 3$ and $x^2 + 5x + 6$; $x^3 - y^3$ and $x^4 + x^2y^2 + y^4$.

7. Find the greatest common divisor of $x^2 - x - 6$ and $x^2 - 4$; $x^2 - 25$ and $x^2 - 2x - 15$.

**83.** The above rule supposes that the quantities may be factored by some one of the methods already explained. The *general* rule depends on the following principle:

*The greatest common divisor of two quantities is the same as that of the less and the remainder, after dividing the greater quantity by the less.* (*Vide* Def. **26**, 1.)

Let $m$ and $n$ be two quantities, $q$ their integral quotient, and $r$ the remainder, and let $d$ be the greatest common divisor of $m$ and $n$, we are to show that $d$ is the greatest common divisor of $n$ and $r$.

From the theory of division we have

$$m = nq + r.$$

Now since $d$ divides $m$ it must divide $nq + r$; and since $d$ divides $n$ it must divide $nq$, and therefore it divides $r$; that is, $d$, the greatest common divisor of $m$ and $n$, is the greatest common divisor of $n$ and $r$.

**84.** To find the greatest common divisor of any two quantities:

(1.) *Divide one quantity by the other, and the last divisor by the last remainder, and so on till there is no remainder.* The last divisor will be the greatest common divisor.

(2.) Any factor common to all the terms of a divisor not found in each term of the corresponding dividend must be rejected.

(3.) When the first term of a dividend is not divisible by the first term of a divisor, multiply this dividend by any quantity that will make its first term exactly divisible by the first term of the divisor.

(4.) The terms of the quotient need not be retained.

(5.) All the signs of any dividend may be at any time changed.

EXAMPLES.

**1.** Find the greatest common divisor of $x^5 - 27x^3 + 22x^2 + 192x - 288$ and $5x^4 - 81x^2 + 44x + 192$.

*Operation.*

| | |
|---|---|
| $x^5 - 27x^3 + 22x^2 + 192x - 288$ | $5x^4 - 81x^2 + 44x + 192$ |
| $5$ | $9$ |
| $5x^5 - 135x^3 + 110x^2 + 960x - 1440$ | $45x^4 - 729x^2 + 396x + 1728$ |
| $5x^5 - 81x^3 + 44x^2 + 192x$ | $45x^4 - 55x^3 - 640x^2 + 1200x$ |
| $-6\,)\,-54x^3 + 66x^2 + 768x - 1440$ | $55x^3 - 89x^2 - 804x + 1728$ |
| $9x^3 - 11x^2 - 128x + 240$ | $9$ |
| $9x^3 + 9x^2 - 108x$ | $495x^3 - 801x^2 - 7236x + 15552$ |
| $-20x^2 - 20x + 240$ | $495x^3 - 605x^2 - 7040x + 13200$ |
| $-20x^2 - 20x + 240$ | $-196\,)\,-196x^2 - 196x + 2352$ |
| $0$ | $x^2 + x - 12$ |

Hence the greatest common divisor is $(x + 4)(x - 3)$.

2. Find the greatest common divisor of $x^3 + 5x^2 + 7x + 3$ and $x^3 + 3x^2 - x - 3$. *Ans.* $x^2 + 4x + 3$.

3. Find the greatest common divisor of $x^3+x^2+2x-4$ and $x^3-8$. *Ans.* $x^2+2x+4$.

4. Find the greatest common divisor of $x^3-8x^2+21x-30$ and $x^4-3x^3+7x^2-3x+6$. *Ans.* $x^2-3x+6$.

5. Find the greatest common divisor of $3x^3+8x^2+8x+5$ and $2x^4+2x^3-5x^2-7x-7$. *Ans.* $x^2+x+1$.

6. Find the greatest common divisor of $x^4-5x^3+9x^2-7x+2$ and $4x^3-15x^2+18x-7$ *Ans.* $(x-1)^2$.

7. Find the greatest common divisor of $x^4-10x^3+37x^2-60x+36$ and $4x^3-30x^2+74x-60$. *Ans.* $(x-3)(x-2)$.

8. Find the greatest common divisor of $x^3-10x^2+33x-36$ and $3x^2-20x+33$.

9. Find the greatest common divisor of $x^3-13x^2+56x-80$ and $3x^2-26x+56$.

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## 85. LEAST COMMON MULTIPLE.

The Least Common Multiple of two or more quantities is the smallest quantity that can be divided by each of them without a remainder. Thus, $xy^2$ is the least common multiple of $xy$ and $y^2$.

To find the least common multiple of two or more quantities when they can be factored:

*Find the product of the smallest collection of factors which includes the factors of each of the given quantities.*

EXAMPLES.

1. Find the least common multiple of $14x^2y^3$ and $28xy^4z^2$. *Ans.* $28x^2y^4z^2$.

2. Find the least common multiple of $4x^2y$, $6x^3y^2$ and $5x^2y^2$. *Ans.* $60x^3y^2$.

3. Find the least common multiple of $x^2-y^2$, $x-y$ and $x+y$. *Ans.* $x^2-y^2$

4. Find the least common multiple of $ax - bx$, $ay - by$ and $xy^3$. *Ans.* $axy^3 - bxy^3$.

5. Find the least common multiple of $a - b$, $a^2 - b^2$, $a + b$ and $x$. *Ans.* $a^2x - b^2x$.

6. Find the least common multiple of $2a^3 - 2b^3$, $3a^2 + 3ab + 3b^2$ and $5x$. *Ans.* $30a^3x - 30b^3x$.

**86.** To find the least common multiple of two quantities which cannot be readily factored:

*Find their greatest divisor; divide one of the quantities by it, and multiply the other by the quotient.*

EXAMPLES.

1. Find the least common multiple of $x^3 + x^2 + 2x - 4$ and $x^3 - 8$. (*Vide* **84**, ex. 3.) *Ans.* $x^4 - x^3 - 8x + 8$.

2. Find the least common multiple of $x^3 - 8x^2 + 21x - 30$ and $x^4 - 3x^3 + 7x^2 - 3x + 6$.
*Ans.* $x^5 - 8x^4 + 22x^3 - 38x^2 + 21x - 30$.

3. Find the least common multiple of $3x^3 + 8x^2 + 8x + 5$ and $2x^4 + 2x^3 - 5x^2 - 7x - 7$.
*Ans.* $6x^5 + 16x^4 - 5x^3 - 46x^2 - 56x - 35$.

4. Find the least common multiple of $x^4 - 5x^3 + 9x^2 - 7x + 2$ and $4x^3 - 15x^2 + 18x - 7$.
*Ans.* $4x^5 - 27x^4 + 71x^3 - 91x^2 + 57x - 14$.

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## *GENERAL REVIEW.*

**87.** The pupil should not be allowed to proceed to the subject of the fractions till he can easily manage the following

EXAMPLES.

1. Add $ax$ to $bx$. *Ans.* $(a + b)x$.

2. Add $x^2 + xy$ to $y^2 + xy$. *Ans.* $(x + y)^2$.

3. Add $(x+y)^2$ to $-2y^2-2xy$. *Ans.* $(x+y)(x-y)$.

4. Add $x^6-10x^3-20$ to $3x^3+12$.

*Ans.* $(x+1)(x^2-x+1)(x-2)(x^2+2x+4.)$

5. From $2x^2+4xy-5y^2$ take $x^2+6xy-6y^2$. *Ans.* $(x-y)^2$.

6. From the sum of $2a-2x+y$, $3a-3x+2y$ and $5a-5x-y$ take $4a-4x+y-2$. *Ans.* $6(a-x)+y+2$.

7. Find the product of $a^2+3(x+1)$ by $x-1$.

*Ans.* $a^2x-a^2+3(x^2-1)$.

8. Multiply $3a^3+3a^2x$ by $a-x$. *Ans.* $3a^2(a^2-x^2)$.

9. Multiply $x^2+bx$ by $x+b$. *Ans.* $x(x+b)^2$.

10. Multiply $x^{\frac{1}{2}}+x^{\frac{1}{2}}y^{\frac{1}{2}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

*Ans.* $x-y+xy^{\frac{1}{2}}-x^{\frac{1}{2}}y$.

11. Divide $1+2x$ by $1-x-x^2$.

*Ans.* $1+3x+4x^2+7x^3+11x^4$ &c.

12. Divide $x$ by $1+x+x^2$.

*Ans.* $x(1-x+x^3-x^4+x^6-x^7+x^9$ &c.

13. Divide $1+x$ by $1-2x+x^2$.

*Ans.* $1+3x+5x^2+7x^3+9x^4$ &c.

14. Divide $x^2+xy+y^2$ by $x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y$. *Ans.* $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

15. Find the factors of $x^3y+2x^2y^2+xy^3$. *Ans.* $xy(x+y)^2$.

16. Find the factors of $x^3y-xy^3$. *Ans.* $xy(x+y)(x-y)$.

17. Find the factors of $x^5y-xy^5$.

*Ans.* $xy(x^2+y^2)(x+y)(x-y)$.

18. From $(a+b)^2$ take $(a-b)^2$. *Ans.* $4ab$.

19. From $x^2(1+x^2+x^4)(x^2-1)$ take $x^2(x^3+1)(x^3-1)$.

*Ans.* 0.

20. From $(x^4+324)\div(x^2+6x+18)$ take $(x-3)(x-2)$.

*Ans.* $-x+12$.

Verify all the above examples when $x=9$, $y=4$, $a=3$, $b=1$, *i. e.* insert these numbers in the given example and reduce; then insert the numbers in the answers, reduce, and see if the results agree.

# CHAPTER IV.

## FRACTIONS.

**88.** An Algebraic *Fraction* represents the quotient of one quantity divided by another. Thus, $\frac{a}{b}$ is a fraction, of which $a$ is the *numerator* and $b$ the *denominator*.

(1.) An *entire quantity* is one not involving a fraction.

(2.) A *mixed quantity* is one uniting an entire and a fractional quantity. Thus, $x + \frac{y}{3}$ is a mixed quantity.

(3.) A *complex fraction* is one whose numerator or denominator contains a fraction. Thus, $\dfrac{a}{\frac{b+c}{x}}$ and $\dfrac{\frac{a+b}{c+d}}{\frac{x+y}{z+w}}$ are complex fractions, and also $\dfrac{1+\frac{x}{y}}{p+q}$.

**89.** All the propositions in arithmetic in relation to fractions are applicable to algebraic fractions.

(1.) The value of a fraction is not changed by multiplying or dividing both terms by the same quantity.

(2.) The value of a fraction is multiplied when the numerator is multiplied or denominator divided.

(3.) The value of a fraction is divided when the numerator is divided or denominator multiplied.

Also the following:

(4.) The value of a fraction is not changed by changing *all* the signs. Thus, $\frac{x^2}{x} = \frac{-x^2}{-x} = x$, and $\frac{-x^2}{x} = \frac{x^2}{-x} = -x$, and $\frac{x^2-y^2}{x-y} = \frac{y^2-x^2}{y-x} = x+y$.

(5.) When a sign is placed before a fraction with a polynomial numerator, the sign does *not* belong to the first term of the numerator but to the whole fraction. Observe, in reducing, the rules of addition and subtraction.

Thus, $+\frac{-a+b-c}{5} = \frac{-a+b-c}{5}$; but $-\frac{-a+b-c}{5} = \frac{a-b+c}{5}$.

**90.** (I.) To reduce a fraction to its lowest terms:

*Divide the numerator and denominator by their greatest common divisor, or cancel the factors common to the numerator and denominator.*

EXAMPLES.

1. Reduce $\frac{17x^2y^2m}{85xy^3m}$ to its lowest terms. *Ans.* $\frac{x}{5y}$.

2. Reduce $\frac{14x^2y^2z^2}{42xy^3z^2}$, $\frac{x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}}{xyz}$ and $\frac{15x^2y^{\frac{1}{2}}z^{\frac{1}{3}}}{105xyz}$ to their lowest terms.

3. Reduce $\frac{5x^2+7ax}{3x+5x^2}$ to its lowest terms. *Ans.* $\frac{5x+7a}{3+5x}$.

4. Reduce $\frac{8x^2+16x^2y}{24x^2+32x^2y}$, $\frac{x^2-y^2}{x^2+2xy+y^2}$ and $\frac{x^2-y^2}{x^2-2xy+y^2}$.

5. Reduce $\frac{x^2-9x+20}{x^2-x-12}$ to its lowest terms. *Ans.* $\frac{x-5}{x+3}$.

6. Reduce $\frac{x^2-2x-35}{x^2+8x+15}$, $\frac{x^2+15x+56}{x^2+5x-14}$ and $\frac{x^2-16}{x^2-x-20}$.

7. Reduce $\frac{x^3y-xy^3}{x^3y-2x^2y^2+xy^3}$, $\frac{x^3-y^3}{x^4+x^2y^2+y^4}$ and $\frac{x^3-y^3}{x^4-y^4}$.

8. Reduce $\frac{27x^3-64y^3}{81x^4+144x^2y^2+256y^4}$. *Ans.* $\frac{3x-4y}{9x^2-12xy+16y^2}$.

9. Reduce $\frac{x^3-4x}{x^3+6x^2+8x}$, $\frac{x^4+2x^3-8x-16}{3(x^2-4)}$, $\frac{x^3+y^3}{x^4+x^2y^2+y^4}$.

10. Reduce $\frac{x^3-8x^2+21x-30}{x^4-3x^3+7x^2-3x+6}$. *Ans.* $\frac{x-5}{x^2+1}$.

11. Reduce $\dfrac{3x^3+8x^2+8x+5}{2x^4+2x^3-5x^2-7x-7}$ and $\dfrac{x^3+5x^2+7x+3}{x^3+3x^2-x-3}$.

(*Vide* **84.**)

(II.) To reduce a fraction to an entire or a mixed quantity:

*Divide the numerator by the denominator, writing the remainder, if there be any, with the denominator under it, at the right of the quotient, with its sign prefixed.*

EXAMPLES.

1. Reduce $\dfrac{12x}{11}$ to a mixed quantity. *Ans.* $x+\dfrac{x}{11}$.

2. Reduce $\dfrac{-27x}{13}$ to a mixed quantity. *Ans.* $-2x-\dfrac{x}{13}$.

3. Reduce $\dfrac{x^2+ax}{x}$ to an entire quantity. *Ans.* $x+a$.

4. Reduce $\dfrac{x^2-a^2}{x}$ to a mixed quantity. *Ans.* $x-\dfrac{a^2}{x}$.

5. Reduce $\dfrac{x^2-9x+20}{x^2-x-12}$ to a mixed quantity. *Ans.* $1-\dfrac{8}{x+3}$.

(*Vide* **90**, (1), 5.)

6. Reduce $\dfrac{3x^2-6a^2}{3x}$, $\dfrac{5x^2+10a^2}{5x}$ and $\dfrac{5x^2+7ax}{5x^2+3x}$.

7. Reduce $\dfrac{x^2-y^2}{x-y}$, $\dfrac{x^3-y^3}{x-y}$, $\dfrac{x^4+y^4}{x+y}$ and $\dfrac{x^4-y^4}{x+y}$.

8. Reduce $\dfrac{x^2+xy+y^2}{x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y}$ and $\dfrac{x^3+x^2y+xy^2+y^3}{x+y}$.

9. Reduce $\dfrac{x^3-4x}{x^3+6x^2+8x}$, $\dfrac{x^3+6x^2+8x}{x^3-4x}$ and $\dfrac{x^2-2x-35}{x^2+8x+15}$.

10. Reduce $\dfrac{x^2+y^2}{x+1}$ to a mixed quantity. *Ans.* $x-1+\dfrac{1+y^2}{1+x}$.

11. Reduce $\dfrac{1+y^2}{1+x}$, $\dfrac{x^2-y^2}{x-1}$, $\dfrac{1-y^2}{x-1}$ and $\dfrac{1-x}{1+x}$.

12. Reduce $\dfrac{x^4 + x^2y^2 + y^4}{x + y}$ and $\dfrac{x^4 - x^2y^2 + y^4}{x - y}$.

13. Reduce $\dfrac{3y^4}{y + x}$, $\dfrac{3x^4}{x + y}$ and $\dfrac{x^3 + 5x^2 + 7x + 3}{x^3 + 3x^2 - x - 3}$.

14. Reduce $\dfrac{x^4 - 3x^3 + 7x^2 - 3x + 6}{x^3 - 8x^2 + 21x - 30}$ *Ans.* $= x + 5 + \dfrac{26}{x - 5}$.

(III.) To reduce an entire or a mixed quantity to the form of a fraction:

(1.) *Multiply the entire quantity by the proposed denominator, and the product will be the numerator;* or,

(2.) *Multiply the entire part by the denominator of the fractional part, and add or subtract the numerator, according to the sign before the fractional part, then place the result over the given denominator.*

EXAMPLES.

1. Reduce $x$ to a fraction whose denominator is 1, 2, or 7.

*Ans.* $\dfrac{x}{1}$, $\dfrac{2x}{2}$, $\dfrac{7x}{7}$.

2. Reduce $x + \dfrac{x}{11}$ to the form of a fraction. *Ans.* $\dfrac{12x}{11}$.

3. Reduce $x - \dfrac{x}{11}$ to the form of a fraction. *Ans.* $\dfrac{10x}{11}$.

4. Reduce $x + \dfrac{1}{11}$ and $x - \dfrac{1}{11}$. *Ans.* $\dfrac{11x + 1}{11}$ and $\dfrac{11x - 1}{11}$.

5. Reduce $a + x + \dfrac{a^2 - x^2}{a - x}$. *Vide* **90**, (I.) *Ans.* $2(a + x)$.

6. Reduce $a - x + \dfrac{a^2 + x^2}{a + x}$, $a - x - \dfrac{a^2 + x^2}{a + x}$, $x + y - \dfrac{x^2 + y^2}{x + y}$.

7. Reduce $x + y - \dfrac{2xy}{x + y}$, $x - y + \dfrac{x^2 + y^2}{x - y}$ and $x - y + \dfrac{2xy}{x - y}$.

8. Reduce $x^2 + xy + y^2 + \dfrac{x^3 + y^3}{x - y}$ and $x^2 - xy + y^2 + \dfrac{x^3 - y^3}{x + y}$.

9. Reduce $x^2 - xy + y^2 - \dfrac{x^2y^2}{x^2+xy+y^2}$, $x^2 - xy + y^2 - \dfrac{x^4+y^4}{x^2+xy+y^2}$.

10. Reduce $1 - x - \dfrac{1+x^2}{1+x}$, $1 + x + x^2 - \dfrac{1+x^3}{1-x}$ and

$$1 + x + x^2 - \frac{1+x^3}{1-x}.$$

11. Reduce $a + 1 - \dfrac{1+a+x}{2}$, $a + x - \dfrac{a^3 - x^3}{a^2 - ax + x^2}$ and

$$1 + 2a - \frac{3a + 2a^2}{1+a}.$$

12. Reduce $5 + \dfrac{x^2 - 16}{x^2 - x - 20}$. (*Vide* **90**, (I), ex. 6.) *Ans.* $\dfrac{6x - 29}{x - 5}$.

13. Reduce $1 - \dfrac{x^2 + 15x + 56}{x^2 + 5x - 14}$ and $2 + \dfrac{x^3 + 6x^2 + 8x}{x^3 - 4x}$.

(IV.) To reduce fractions having different denominators to equivalent fractions having a *least common denominator*:

(1.) *Reduce the fractions to their lowest terms,* unless they are so given as to admit of no reduction.

(2.) *Find the least common multiple of all the denominators.*

(3.) *Divide this multiple by the denominator of the first reduced fraction, and multiply the quotient by the numerator, and write the product over the multiple.* Do the same for all the fractions, and the resulting fractions will be those required.

EXAMPLES.

1. Reduce $\dfrac{x}{x^2 + xy}$, $\dfrac{2ax}{x^3 - xy^2}$ and $\dfrac{x + x^2}{x^2 - xy}$ to equivalent fractions having a least common denominator.

*Solution.*

$$\frac{x}{x^2+xy},\ \frac{2ax}{x^3-xy^2},\ \frac{x+x^2}{x^2-xy}, \qquad = \text{fractions given.}$$

$$\frac{1}{x+y},\ \frac{2a}{x^2-y^2},\ \frac{1+x}{x-y}, \qquad = \text{fractions reduced.}$$

$$\frac{x-y}{x^2-y^2}, \quad \frac{2a}{x^2-y^2}, \quad \frac{x+x^2+y+xy}{x^2-y^2} = \text{fractions required.}$$

2. Reduce $x+10$, $\frac{x^2+4x+3}{x^2+x-6}$ and $\frac{x^2+8x+15}{x^2-25}$ to equivalent fractions having a least common denominator.

*Solution.*

$$\frac{x+10}{1}, \quad \frac{x^2+4x+3}{x^2+x-6}, \quad \frac{x^2+8x+15}{x^2-25} = \text{fractions given.}$$

$$\frac{x+10}{1}, \frac{(x+3)(x+1)}{(x+3)(x-2)}, \frac{(x+5)(x+3)}{(x+5)(x-5)} = \text{fractions factored.}$$

$$\frac{x+10}{1}, \quad \frac{x+1}{x-2}, \quad \frac{x+3}{x-5} = \text{fractions reduced.}$$

$$\frac{x^3+3x^2-60x+100}{x^2-7x+10}, \quad \frac{x^2-4x-5}{x^2-7x+10}, \quad \frac{x^2+x-6}{x^2-7x+10} \text{ fractions required.}$$

3. Reduce $\frac{x}{2}, \frac{x}{3}$ and $\frac{x}{4}$. *Ans.* $\frac{6x}{12}, \frac{4x}{12}$ and $\frac{3x}{12}$.

4. Reduce $\frac{1}{x}, \frac{1}{y}$ and $\frac{1}{z}$. *Ans.* $\frac{yz}{xyz}, \frac{xz}{xyz}$ and $\frac{xy}{xyz}$.

5. Reduce $\frac{1}{x}, \frac{1}{x^2}$ and $\frac{1}{x^3}$. *Ans.* $\frac{x^2}{x^3}, \frac{x}{x^3}$ and $\frac{1}{x^3}$.

(v.) To add fractional quantities together:

(1.) *Reduce the fractions to the least common denominator.*

(2.) *Add the numerators, placing the sum over the least common denominator.*

In mixed quantities, add the entire parts first, to which annex the sum of the fractional parts by the proper sign.

EXAMPLES.

1. Add the quantities $x+\frac{1}{x+y}$, $3x-\frac{1}{x^2-y^2}$, $-2x+\frac{1}{x-y}$.

*Solution.*

The sum of the entire parts is $x + 3x - 2x = 2x$.

$$\frac{1}{x+y}, \quad \frac{-1}{x^2-y^2}, \quad \frac{1}{x-y} \quad = \text{fractions given.}$$

$$\frac{x-y}{x^2-y^2}, \quad \frac{-1}{x^2-y^2}, \quad \frac{x+y}{x^2-y^2} \quad = \text{fractions reduced. (IV.)}$$

$$2x + \frac{2x-1}{x^2-y^2} \quad = \text{the sum as above.}$$

2. Add the fractions $\frac{x}{2}$, $\frac{x}{3}$, $\frac{x}{4}$ and $\frac{x}{5}$. *Ans.* $\frac{77x}{60} = x + \frac{17x}{60}$. (II.)

3. Find the sum of $\frac{x}{11} + \frac{x}{13} + \frac{x}{26} + \frac{x}{52}$.

4. Find the sum of $\frac{1}{x} + \frac{1}{y}$, also of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, also of $\frac{1}{x} + \frac{1}{x^2}$.

5. Find the sum of $\frac{1}{x^2} + \frac{1}{x^3}$, also of $\frac{2}{x^2y} + \frac{3}{xy^2} + \frac{5}{x^2y^2}$.

6. Find the sum of $\frac{x+1}{x^2}$ and $\frac{x+1}{x^3}$. *Ans.* $\frac{(x+1)^2}{x^3}$.

7. Add the fractions $\frac{1}{x+y}$, $\frac{2}{x^2-y^2}$ and $\frac{3}{x-y}$.
*Ans.* $\frac{2(2x+y+1)}{x^2-y^2}$

8. $\frac{1}{x^3-y^3} + \frac{x+y}{x^2+xy+y^2} + \frac{1}{x-y}$.

9. $\frac{1}{x^4-y^4} + \frac{1}{x^3+x^2y+xy^2+y^3} + \frac{1}{x^2+y^2} + \frac{1}{x^2-y^2}$.

10. $\frac{1}{x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y} + \frac{1}{x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y}$.

11. $\frac{4x}{x-2} + \frac{x+2}{x+3}$. *Ans.* $5 + \frac{7x+26}{x^2+x-6}$.

12. $\frac{x+3}{x-7} + \frac{x-7}{x+3}$.

13. $\frac{x+4}{x-5} + \frac{x+5}{x-4}$.

14. $\dfrac{x^2+4x+3}{x^2+x-6} + \dfrac{x^2+8x+15}{x^2-25}$.

15. $\dfrac{x^2-1}{x^2+6x-7} + \dfrac{x^2-36}{x^2-2x-48}$.

16. $\dfrac{x^2+5x+4}{x^2+3x-4} + \dfrac{x^2+4x-5}{x^2+6x+5}$.

17. $\dfrac{x^3+x^2+x+1}{x^3+x^2-x-1} + \dfrac{x^3-x^2-x+1}{x^3-x^2+x-1}$.

18. $\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y}$.

19. $\dfrac{x-y}{x^3-y^3} + \dfrac{x+y}{x^3+y^3}$.

20. $\dfrac{x-y}{x^5-y^5} + \dfrac{x+y}{x^5+y^5}$.

21. $\dfrac{2x}{1-x^2} + \dfrac{1}{x+1}$.

22. $\dfrac{2}{(x-1)^3} + \dfrac{3}{(x-1)^2} + \dfrac{4}{x-1}$.

23. $\dfrac{x}{(1+x)(x+y)} + \dfrac{y}{(1-y)(x+y)}$.

24. $\dfrac{1}{4(1+x)} + \dfrac{1}{4(1-x)} + \dfrac{1}{2(1-x^2)}$.

25. Add $7x + \dfrac{y}{7} - 51$ to $\dfrac{x}{7} + 7y - 99$.

26. Add $\dfrac{3x}{4} + \dfrac{2y}{5} - \dfrac{61}{120}$ to $\dfrac{2x}{5} + \dfrac{3y}{4} - \dfrac{9}{20}$.

*Ans.* to last $\dfrac{23x}{20} + \dfrac{23y}{20} - \dfrac{23}{24}$.

(VI.) To subtract one fraction from another:

(1.) *Reduce the fractions to the least common denominator.*

(2.) *Subtract the numerator of the subtrahend from that of the minuend, placing the difference over the least common denominator.*

Mixed quantities may be reduced to a fractional form, or the parts subtracted separately.

EXAMPLES.

1. From $3x + \frac{x}{2}$ take $x - \frac{4x}{5}$.

*Operation.*

$$3x + \frac{x}{2}. \qquad x - \frac{4x}{5}. \qquad = \text{quantities given.}$$

$$\frac{7x}{2}. \qquad \frac{x}{5}. \qquad = \text{quantities reduced. (II.)}$$

$$\text{Then } \frac{35x}{10} - \frac{2x}{10} = \frac{33x}{10} = 3x + \frac{3x}{10}. \quad \textit{Ans.}$$

2. From $\frac{x^2 + 6x + 8}{x^2 - x - 6}$ take $\frac{x^2 + 3x - 40}{x^2 + x - 30}$.

*Operation.*

$$\frac{(x+2)(x+4)}{(x+2)(x-3)} \qquad \frac{(x+8)(x-5)}{(x+6)(x-5)} \qquad = \text{fractions factored.}$$

$$\frac{x+4}{x-3} \qquad \frac{x+8}{x+6} \qquad = \text{factors cancelled.}$$

$$\frac{x^2+10x+24}{x^2+3x-18} \qquad \frac{x^2+5x-24}{x^2+3x-18} \qquad = \text{fractions reduced. (IV.)}$$

$$\textit{Ans. } \frac{5x+48}{x^2+3x-18}.$$

3. From $\frac{1}{x}$ take $\frac{1}{y}$. *Ans.* $\frac{y-x}{xy}$.

4. From $3x + \frac{1}{x}$ take $x + \frac{1}{y}$.

5. From $7x$ take $\frac{4x}{5}$.

6. From $\frac{1}{x}$ take $\frac{1}{x^2}$.

7. From $\frac{x+1}{x-1}$ take $\frac{x-1}{x+1}$.

8. From $\frac{x+y}{x-y}$ take $\frac{x-y}{x+y}$.

9. From $\frac{x+4}{x-6}$ take $\frac{x-2}{x+3}$.

10. From $\frac{x^2+4x-12}{x^2+5x-14}$ take $\frac{x^2+2x-15}{x^2+3x-18}$.

11. From $\frac{x^2-9}{x^2+x-12}$ take $\frac{x^2+x-20}{x^2-25}$.

12. From $\frac{x+y}{x^3+y^3}$ take $\frac{x-y}{x^3-y^3}$. 13. From $\frac{x^2+1}{x^2-1}$ take $\frac{x^2-1}{x^2+1}$.

14. From $\frac{x+y}{x-y}$ take $\frac{x^2-2xy+y^2}{x^2-y^2}$.

15. From $\frac{1}{x+1}$ take $\frac{x-2}{x^2-x+1}$.

16. From $\frac{1}{y}+\frac{1}{z}$ take $\frac{1}{y}-\frac{1}{z}$. *Ans.* $\frac{2}{z}$.

17. From $m$ take $\frac{bm+c}{a+b}$.

18. From $\frac{ab+ac+bc}{2abc}$ take $\frac{1}{c}$, $\frac{1}{b}$ and $\frac{1}{a}$ respectively.

19. From $2b^3-\frac{1+a}{3}$ take $b^2-\frac{1-a}{4}$.

20. From $3x+\frac{11x-10}{15}$ take $2x+\frac{3x-5}{7}$.

(VII.) To multiply fractional quantities:

(1.) *Reduce mixed quantities to the form of a fraction.* (III.)

(2.) *Factor each numerator and denominator, and cancel such factors as are found in both numerator and denominator.*

(3.) *Multiply the remaining factors of the numerators for a new numerator, and those of the denominator for a new denominator.*

(4.) *Reduce the result, if necessary, to a mixed quantity.* (II.)

EXAMPLES.

1. Multiply $a+\frac{ax}{a-x}$ by $\frac{a^2-x^2}{x+x^2}$.

*Operation.*

$$a + \frac{ax}{a - x}. \qquad \frac{a^2 - x^2}{x + x^2}. \qquad = \text{given quantities.}$$

By II. $$\frac{a^2}{a - x}. \qquad \frac{(a + x)\ (a - x)}{x\,(1 + x)}. \qquad = \text{quantities factored.}$$

Then $$\frac{a^2\,(a + x)}{x\ (1 + x)}. \qquad = \text{result required.}$$

2. Multiply $\dfrac{x^4 - y^4}{x^2 - 2xy + y^2}$ by $\dfrac{x - y}{x^2 + xy}$.

*Operation.*

$$\frac{x^4 - y^4}{x^2 - 2xy + y^2}. \qquad \frac{x - y}{x^2 + xy}. \qquad = \text{given quantities.}$$

$$\frac{(x^2 + y^2)\ (x + y)\ (x - y)}{(x - y)\ (x - y)}. \qquad \frac{x - y}{x\,(x + y)} \qquad = \text{quantities factored.}$$

Then $$\frac{x^2 + y^2}{x} = x + \frac{y^2}{x} \qquad = \text{quantity required.}$$

3. Multiply $\dfrac{a^2 bx}{3mn}$ by $\dfrac{6m^2 n^2}{a^4 bx}$. *Ans.* $\dfrac{2mn}{a^2}$.

4. Multiply $\dfrac{x + 1}{x - 1}$ by $\dfrac{x - 1}{x + 1}$. *Ans.* 1.

5. Multiply $2 + \dfrac{2}{x - 1}$ by $2 - \dfrac{2}{x + 1}$.

6. $2 + \dfrac{2}{x - 1}$ by $\dfrac{2}{x + 1}$

7. $\dfrac{x^2 - y^2}{x^2 + 2xy + y^2}$ by $\dfrac{x + y}{x - y}$.

8. $\dfrac{x^4 - y^4}{x^2 - 2xy + y^2}$ by $\dfrac{x - y}{x^2 + y^2}$.

9. $\dfrac{x^3 + y^3}{x^2 - y^2}$ by $\dfrac{x + y}{x^2 - xy + y^2}$.

10. $\dfrac{x^6 - y^6}{x^4 - y^4} \times \dfrac{x^2 + y^2}{x^2 + xy + y^2} \times \dfrac{x + y}{x^2 - xy + y^2} =$

11. $\dfrac{x^2 - x - 2}{x^2 - x - 12} \times \dfrac{x^2 + x - 20}{x^2 - 6x - 7} =$

12. $\left(1-\frac{x-19}{x^2-25}\right)\times\left(1-\frac{2x+11}{x^2-4}\right)=$

13. $\left(1+\frac{10x-5}{x^2-6x+9}\right)\times\left(1-\frac{5}{x^2-4}\right)=$

14. $\left(1-\frac{14x+7}{x^2+8x+16}\right)\times\left(1-\frac{7}{x^2-9}\right)=$

15. $\left(1-\frac{3ax-a^2}{x^2+2ax-3a^2}\right)\times\left(1-\frac{5ax+15a^2}{x^2+ax-6a^2}\right)$.

16. $\left(1+\frac{a+x}{a-x}\right)\times\left(1-\frac{a-x}{a+x}\right)$.

17. $\left(1+x+\frac{2+x^2}{1-x}\right)\times\left(1+2x-\frac{3-4x^2}{1-2x}\right)$.

18. $\left(a-x+\frac{a^2+x^2}{a+x}\right)\times\left(a-x-\frac{a^2+x^2}{a+x}\right)$.

19. $\left(x^2+xy+y^2+\frac{x^3+y^3}{x-y}\right)\times\left(x^2-xy+y^2+\frac{x^3-y^3}{x+y}\right)$

20. $(x^2+x+1)\times\left(\frac{1}{x^2}-\frac{1}{x}+1\right)$.

21. $\left(x+1+\frac{1}{x}\right)\left(x-1+\frac{1}{x}\right)$.

22. $(a+b)\times\frac{c+d}{a+b}=c+d$.

(VIII.) To divide fractional quantities:

(1.) *Reduce mixed quantities to the form of a fraction.*

(2.) *Invert the divisor, and then proceed as in multiplication.*

EXAMPLES.

1. Divide $1-x-\frac{1+x^2}{1+x}$ by $1+x-x^2-\frac{1+x^3}{1-x}$.

*Operation.*

$$1 - x - \frac{1+x^2}{1+x} = \frac{-2x^2}{1+x}. \qquad (\textit{Vide } \text{III.})$$

$$1 + x - x^2 - \frac{1+x^3}{1-x} = \frac{-2x^2}{1-x}. \qquad (\textit{Vide } \text{III.})$$

$$\text{Then } \frac{-2x^2}{1+x} \times \frac{1-x}{-2x^2} = \frac{1-x}{1+x} \quad \textit{Ans.}$$

2. Divide $\frac{7xy^2}{4z}$ by $\frac{4xy^2z}{3}$. *Ans.* $\frac{21}{16z^2}$.

3. $\left(x + \frac{x}{11}\right) \div \left(x - \frac{x}{11}\right) = \quad 1\frac{1}{5}$.

4. $\frac{x+y}{x-y} \div \frac{x+y}{4}$.

5. $\left(a - x + \frac{a^2+x^2}{a+x}\right) \div \left(a - x - \frac{a^2+x^2}{a+x}\right)$.

6. $\left(1 - x - \frac{1+x^2}{1+x}\right) \div \left(1 + x + x^2 - \frac{1+x^3}{1-x}\right)$.

7. $\frac{x^2-4}{x^2-16} \div \frac{x^2-9x+14}{x^2+2x-8}$. *Ans.* $\frac{x^2-4}{x^2-11x+28}$.

8. $\frac{1-x^2}{1+5x+6x^2} \div \frac{1+3x+2x^2}{1+x-6x^2}$. *Ans.* $\frac{1-3x+2x^2}{1+4x+4x^2}$.

9. $\frac{a^4-x^4}{a^2+2ax+x^2} \div \frac{a-x}{a+x}$. *Ans.* $a^2+x^2$.

10. $\frac{x^3-y^3}{x^2+y^2} \div \frac{x^2+xy+y^2}{x^4-y^4}$. *Ans.* $x^3 - x^2y - xy^2 + y^3$.

11. $\frac{x^2+2xy+y^2}{x^2-y^2} \div \frac{x+y}{x^2-2xy+y^2}$. *Ans.* $x-y$.

12. $\left(1 + \frac{x-1}{x+1}\right) \div \left(1 - \frac{x-1}{x+1}\right)$. *Ans.* $x$.

13. $\left(1 + \frac{x+1}{x-1}\right) \div \left(1 - \frac{x-1}{x+1}\right)$. *Ans.* $\frac{x(x+1)}{x-1}$.

14. $\left(1-x+\dfrac{1+x^2}{1+x}\right) \div \left(1-x-\dfrac{1+x^2}{1+x}\right)$.

15. $\left(1+x+\dfrac{1+x^2}{1-x}\right) \div \left(1-x-\dfrac{1+x^2}{1+x}\right)$.

16. $\left(x^2-xy+y^2-\dfrac{x^2y^2}{x^2+xy+y^2}\right) \div \left(x^2-xy+y^2-\dfrac{x^4+y^4}{x^2+xy+y^2}\right)$.

17. $\left(x+y-\dfrac{2xy}{x+y}\right) \div \left(x+y-\dfrac{x^2+y^2}{x+y}\right)$.

18. $\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}-\dfrac{x+y}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right) \div \left(x^{\frac{1}{2}}+y^{\frac{1}{2}}-\dfrac{2x^{\frac{1}{2}}y^{\frac{1}{2}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right)$.

19. $\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right) \div \left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$.

20. $\left(\dfrac{x^2-y^2}{x^2+y^2}+\dfrac{x^2+y^2}{x^2-y^2}\right) \div \left(\dfrac{x^2-y^2}{x^2+y^2}-\dfrac{x^2+y^2}{x^2-y^2}\right)$.

21. $\left(\dfrac{1}{x}+\dfrac{1}{x^2}\right) \div \left(\dfrac{1}{x^2}+\dfrac{1}{x^3}\right)$. *Ans.* $x$.

22. $\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right) \div \left(\dfrac{x-y}{x^3-y^3}+\dfrac{x+y}{x^3+y^3}\right)$.

23. $\left(\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}\right) \div \dfrac{x^2}{x-1}$. *Ans.* $\dfrac{4}{x+1}$.

24. $\left(\dfrac{x+y}{x-y}-\dfrac{x^2-2xy+y^2}{x^2-y^2}\right) \div \left(\dfrac{1}{x+y}+\dfrac{3}{x^2-y^2}+\dfrac{3}{x-y}\right)$.

25. Divide 1 by $\dfrac{ab+bc+ac}{2abc}-\dfrac{1}{c}$.

26. Divide 1 by $\dfrac{ab+bc+ac}{2abc}-\dfrac{1}{b}$.

27. Divide 1 by $\dfrac{ab+bc+ac}{2abc}-\dfrac{1}{a}$.

28. $\left(x^4-\dfrac{1}{x^4}\right) \div \left(x-\dfrac{1}{x}\right)$. *Ans.* $x^3+x+\dfrac{1}{x}+\dfrac{1}{x^3}$.

## *REVIEW OF FRACTIONS.*

**91.** Perform the operations indicated in the following

EXAMPLES.

1. $\frac{x-1}{3}+\frac{x-4}{5}+\frac{x-7}{10}+\frac{11x+55}{30}$. *Ans.* $x$.

2. $\frac{x^6}{x^4-1}-\frac{x^4}{x^2+1}+\frac{1}{x^2+1}-\frac{1}{x^2-1}$. *Ans.* $\frac{x^4-2}{x^4-1}$.

3. $\left(1+\frac{x+1}{x-1}\right)\times\left(1+\frac{x-1}{x+1}\right)\div\left(1+\frac{x+1}{x-1}\right)\times\left(1-\frac{x-1}{x+1}\right)$.

*Ans.* $x$.

4. $\frac{x+1}{3x^2-11x+6}+\frac{x-1}{2x^2-7x+3}+\frac{x-2}{6x^2-7x+2}$.

*Ans.* $\frac{6x^2-9x+7}{6x^3-25x^2+23x-6}$.

5. $\frac{3x^3+8x^2+8x+5}{2x^4+2x^3-5x^2-7x-7}-\frac{x^3+5x^2+7x+3}{x^3+3x^2-x-3}$.

*Ans.* $\frac{2x^3+5x^2-5x-12}{2x^3-2x^2-7x+7}$.

6. $\frac{2x-14}{x^2-3x-28}+\frac{3x-21}{x^2-11x+28}-\frac{4x-28}{x^3-7x^2-16x+112}$.

*Ans.* $\frac{5x}{x^2-16}$.

7. $\frac{9+y^2}{2}\times\frac{9-y^2}{5}\div\frac{3-y}{10}$. *Ans.* $27+9y+3y^2+y^3$.

8. $\frac{3}{2x-3}-\frac{2x-15}{4x^2+9}-\frac{2}{2x+3}$. *Ans.* $\frac{12x(10x+3)}{16x^4-81}$.

9. $\frac{1}{x^4+4a^4}+\frac{2}{x^2+2ax+2a^2}-\frac{2}{x^2-2ax+2a^2}$.

*Vide* **61**, ex. 36. *Ans.* $\frac{1-8ax}{x^4+4a^4}$.

# CHAPTER V.

## 92. EQUATIONS OF THE FIRST DEGREE.

(1.) An *equation* is an algebraic expression showing that two quantities are equal. (*Vide* Def. **15**, also Def. **3** and **4**.)

(2.) An equation is of the *first degree* when its unknown quantity is involved to the *first power* only. (*Vide* Def. **13**.)

(3.) A *numerical equation* contains only numbers and the unknown quantity. Thus, $3x + 5x = 30$.

(4.) A *literal equation* contains letters representing known quantities. Thus, $3x + ax = b$, where $a$ represents 5 and $b$ 30 of the equation in (3.)

(5.) An *identical equation* is one in which both members are alike, or in which either member is the result of operations indicated by the other. Thus, $\frac{x+1}{4+1} = \frac{1+x}{5}$, and $\frac{1+x}{1-x} = 1 + 2x + 2x^2 + \&c.$

(6.) An equation is *verified* when on the substitution of a quantity for $x$, it is rendered identical. Thus, if for $x$ in the equation $x + 3x = 24$, the number 6 is substituted, it becomes $6 + 18 = 24$, where 6 is the *only* number which will render the given equation identical.

(7.) The *solution* of an equation consists in finding the quantity which will verify it. This quantity is called a *root* of the equation.

(8.) The solution of an equation depends upon one or more of the following self-evident propositions, called axioms.

**93.** AXIOMS.

(1.) Quantities which are equal to the same thing are equal to each other.

(2.) If to equal quantities equal quantities be added, the sums will be equal.

(3.) If from equal quantities equal quantities be subtracted, the remainders will be equal.

(4.) If equal quantities be multiplied by the same or equal quantities, the products will be equal.

(5.) If equal quantities be divided by the same or equal quantities, the quotients will be equal.

(6.) If quantities are equal, their like roots are equal. (*Vide* Def. **13**, 3.)

(7.) If quantities are equal, their like powers are equal. (*Vide* Def. **13**, 2.)

## SOLUTION OF EQUATIONS OF THE FIRST DEGREE CONTAINING ONE UNKNOWN QUANTITY.

**94.** The object of every change in equations is, ultimately, to make the unknown quantity $x$ constitute the first member, and the known quantities, reduced to their simplest form, the second member. The equation is then solved. (*Vide* **92**, (7.)

**95.** To solve an equation of the form $ax = b$.

Since $ax = b$, by axiom (5) $x = \frac{b}{a}$. Hence,

Divide the equation by the coëfficient of $x$.

EXAMPLES.

1. Solve the equation $2x = 10$.

*Solution.*

$2x = 10$ (1) = given equation.

$x = 5$ (2) = required equation.

Equation (2) is obtained by dividing (1) by the number 2, which is the coëfficient of $x$. The number 5 will verify the given equation, for $2 \times 5 = 10$.

In the same way, solve and verify the equations

2. $7x = 21$. *Ans.* $x = 3$.
3. $5x = 25$.
4. $4x = 144$.
5. $3x = 15$.
6. $10x = 20$.
7. $9x = 729$.
8. $12x = 156$.
9. $13x = 169$.
10. $ax = a^2$. *Ans.* $x = a$.
11. $bx = ab + b^2$. *Ans.* $x = a + b$.
12. $ax = 5b$. *Ans.* $x = \frac{5b}{a}$
13. $2ax = c$. *Ans.* $x = \frac{c}{2a}$.

**96.** If both members consist of several terms;

*Unite these terms* (by **33**) *and then divide by the coëfficient of* $\boldsymbol{x}$.

EXAMPLES.

14. Solve the equation $2x + 3x = 30 + 15$.

*Solution.*

$2x + 3x = 30 + 15$ (1) = given equation.

$5x = 45$ (2) (*Vide* **33**.)

$x = 9$ (3) = (2) ÷ 5. Axiom (5.)

$2 \times 9 + 3 \times 9 = 30 + 15$ = verification.

15. Solve the equation $5x - 2x = 50 - 20$. *Ans.* $x = 10$.
16. Solve the equation $5x + 3x - 2x = 50 - 40 + 2$. *Ans.* $x = 2$.
17. Solve the equation $20x - 18x + 4x = 100 - 70 + 30$. *Ans.* $x = 10$.
18. Solve the equation $11x + 15x - 10x = 100 - 10 + 14 - 8$. *Ans.* $x = 6$.
19. Solve the equation $ax + bx = c + d$.

*Solution.*

$ax + bx = c + d$ (1) = given equation.

$(a + b)\,x = c + d$ (2) (*Vide* **75**, ex. 3.)

$$x = \frac{c + d}{a + b}.$$

Then, $a \times \dfrac{c + d}{a + b} + b \times \dfrac{c + d}{a + b} = c + d$ = verification.

If $a = 2$, $b = 3$, $c = 30$ and $d = 15$, then (ex. 14) $x = 9$.

If $a = 5$, $b = -2$, $c = 50$ and $d = -20$, then (ex. 15) $x = 10$.

20. Solve the equation $ax + bx - cx = d - e + f$.

*Ans.* $x = \dfrac{d - e + f}{a + b - c}$.

If $a = 5$, $b = 3$, $c = 2$, $d = 50$, $e = 40$ and $f = 2$, then $x = 2$.

If $a = 6$, $b = 4$, $c = 5$, $d = 40$, $e = 20$ and $f = 10$, then $x = 6$.

**97.** To solve an equation of the form $\dfrac{x}{a} = b$.

Since $\dfrac{x}{a} = b$, by axiom (4) $x = ab$. Hence,

Multiply the equation by the denominator of the fraction.

EXAMPLES.

1. Solve the equation $\dfrac{x}{3} = 7$.

*Solution.*

$\dfrac{x}{3} = 7$ (1) = given equation.

$x = 21$ (2) = required equation.

$\dfrac{21}{3} = 7$ = verification.

Equation (2) is obtained by multiplying (1) by the number 3.

In the same manner solve and verify the following.

2. $\dfrac{x}{5} = 20$. *Ans.* $x = 100$.

3. $\dfrac{2x}{3} = 6$. *Ans.* $x = 9$.

4. $\dfrac{4x}{9} = 16$.

5. $\dfrac{3x}{5} = 15$.

6. $\frac{7x}{9} = 14.$ *Ans.* $x = 18.$

7. $\frac{9x}{12} = 18.$ *Ans.* $x = 24.$

8. $\frac{11x}{13} = 22.$

9. $\frac{13x}{21} = 39.$

10. $\frac{ax}{3} = a^2.$ *Ans.* $x = 3a.$

11. $\frac{4bx}{c} = a.$ *Ans.* $x = \frac{ac}{4b}.$

12. $\frac{3bx}{5c} = 26.$ *Ans.* $x = \frac{130c}{3b}.$

13. $\frac{x}{a+b} = c.$ *Ans.* $x = (a+b)c.$

**98.** When there are several fractions:

*Multiply the equation by the least common multiple of all the denominators*, after which proceed as in **96.**

EXAMPLES.

14. Solve the equation $\frac{x}{2} + \frac{x}{3} = 10.$

*Solution.*

$$\frac{x}{2} + \frac{x}{3} = 10 \qquad (1) = \text{given equation.}$$

$$3x + 2x = 60 \qquad (2)$$

$$5x = 60 \qquad (3) \qquad (\textit{Vide } \mathbf{96.})$$

$$x = 12 \qquad (4) = \text{required equation.}$$

$$\frac{12}{2} + \frac{12}{3} = 10 \qquad = \text{verification.}$$

Equation (2) is obtained by multiplying both sides of (1) by 6, the least common multiple of 2 and 3.

15. Solve the equation $\frac{x}{2} + \frac{4x}{3} - \frac{3x}{4} = 13.$

*Solution.*

$$\frac{x}{2} + \frac{4x}{3} - \frac{3x}{4} = 13 \qquad (1) = \text{given equation.}$$

$$6x + 16x - 9x = 156 \qquad (2) = (1) \times 12.$$

$$13x = 156 \qquad (3)$$

$$x = 12 \qquad (4) = \text{required equation.}$$

$$\frac{12}{2} + \frac{4 \times 12}{3} - \frac{3 \times 12}{4} = 13 \qquad = \text{verification.}$$

In the same manner solve the equations,

16. $\frac{3x}{4} + \frac{2x}{5} = 23.$ *Ans.* $x = 20.$

17. $x + \frac{x}{2} + \frac{x}{3} = 11.$ *Ans.* $x = 6.$

18. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = 7.$ *Ans.* $x = 12.$

19. $\frac{2x}{5} + \frac{4x}{9} - \frac{x}{45} = 37.$ *Ans.* $x = 45.$

20. $\frac{3x}{7} - \frac{4x}{21} + \frac{5x}{42} = 15.$ *Ans.* $x = 42.$

21. $\frac{4x}{5} - \frac{3x}{7} + \frac{2x}{35} = 15.$ *Ans.* $x = 35.$

22. $\frac{x}{3} + 6x = 38.$ *Ans.* $x = 6.$ 23. $\frac{x}{a} + \frac{x}{b} = c.$

*Solution.*

$$\frac{x}{a} + \frac{x}{b} = c \qquad (1) = \text{given equation.}$$

$$bx + ax = abc \qquad (2) = (1) \times ab.$$

$$(a + b)\,x = abc \qquad (3) \qquad (\textit{Vide } 75, \text{ ex. } 3.)$$

$$x = \frac{abc}{a + b} \qquad (4) = \text{required equation.}$$

If $a = 2$, $b = 3$ and $c = 10$, then (ex. 14) $x = 12$.
If $a = 3$, $b = 4$ and $c = 14$, then $x = 24$.
If $a = 5$, $b = 6$ and $c = 11$, then $x = 30$.
If $a = 7$, $b = 4$ and $c = 22$, then $x = 56$.

24. Solve the equation $\frac{mx}{a} + \frac{nx}{b} = c.$ *Ans.* $x = \frac{abc}{bm + an}.$

If $m = 3$, $n = 2$, $a = 4$, $b = 5$ and $c = 23$, then (ex. 16) $x = 20$.

**99.** To solve the equation $ax + d = c - bx$.

By axiom (3) we may subtract $d$ from both sides of the given equation; thus: $$ax + d - d = c - bx - d. \qquad (2)$$

In the first member $+d$ now cancels $-d$, giving the equation $$ax = c - bx - d. \qquad (3)$$

By axiom (2) we may now add $bx$ to both sides of equation (3); thus: $$ax + bx = c - bx + bx - d. \qquad (4)$$

In the second member $-bx$ cancels $+bx$, giving the equation $$ax + bx = c - d. \qquad (5)$$

If we now compare (5) with the given equation, we see that $-bx$ has been transposed to the first member of (5), its sign being changed to $+$. Also, $+d$, of the given equation, has been transposed to the second member of (5), its sign being changed to $-$. The value of $x$ is now found by **96**, and is

$$x = \frac{c - d}{a + b}. \qquad (6)$$

Hence,

*Transpose the terms involving* x *to the first member, changing the signs.*

*Transpose the terms not involving* x *to the second member, changing the signs.* After this proceed as in **96.**

EXAMPLES.

1. Solve the equation $9x - 5 = 40$.

*Solution.*

| | | | |
|---|---|---|---|
| | $9x - 5 = 40$ | (1) | = given equation. |
| Transpose $-5$, and we have | $9x = 40 + 5$ | (2) | |
| | $9x = 45$ | (3) | by **96.** |
| | $x = 5$ | (4) | = required equation. |

2. Solve the equation $10 + 3x - 20 = x + 50$.

*Solution.*

$$10 + 3x - 20 = x + 50 \quad (1)$$
$$3x - x = 50 - 10 + 20 \quad (2)$$
$$2x = 60 \quad (3)$$
$$x = 30 \quad (4)$$

3. Solve the equation $15 + x - 20 = 5x - 7x + 40$. *Ans.* $x = 15$.

4. Solve the equation $12x - 50 + 60 - 15x = 8x - 70$.
*Ans.* $x = 7\frac{3}{11}$.

5. Solve the equation $14x - 20 + 5 = 16x + 25 - 50$.
*Ans.* $x = 5$.

6. Solve the equation $ax + c = bx + d$. *Ans.* $x = \dfrac{d - c}{a - b}$.
If $a = 20$, $b = 10$, $c = 60$ and $d = 80$, then $x = 2$.
If $a = 5$, $b = 4$, $c = 7$ and $d = 8$, then $x = 1$.

7. Solve the equation $x + 5a = 2b - cx$. *Ans.* $x = \dfrac{2b - 5a}{1 + c}$.
If $b = 7$, $a = 1$, $c = 2$, then $x = 3$.
If $b = 9$, $a = 2$, $c = 3$, then $x = 2$.

**100.** Hence, to solve an equation of the first degree with one unknown quantity:

(1.) *Clear the equations of fractions by multiplying by the least common multiple of all the denominators.* (By **98.**)

(2.) *Transpose the terms involving* x *to the first member, and those not involving* x *to the second member.* (By **99.**)

(3.) *Unite the terms of the first member so as to indicate a single coëfficient of* x (by **96**), *and reduce the terms of the second member to as simple a form as possible.*

(4.) *Divide the equation by the coëfficient of* x.

EXAMPLES.

1. Solve the equation $\dfrac{x}{2} - \dfrac{x + 1}{4} = \dfrac{9}{2}$.

*Solution.*

$$2x - x - 1 = 18 \qquad (1) \quad \textit{Vide } \mathbf{89},\ (5).$$

$$x = 19 \qquad (2)$$

Verification $\dfrac{19}{2} - \dfrac{19+1}{4} = \dfrac{9}{2}$.

2. Solve the equation $x + 5x - 20 = 3x + 80$. *Ans.* $x = 33\frac{1}{3}$.

3. Solve the equation $x - \dfrac{x-3}{2} + \dfrac{7x}{5} - 20 = 4x - 29$.

*Ans.* $x = 5$.

Solve the following equations, and verify the result.

4. $x + \dfrac{x}{2} + \dfrac{x}{3} + \dfrac{x}{4} = 12\dfrac{1}{2}$.

5. $x - \dfrac{x}{2} - \dfrac{x}{3} - \dfrac{x}{4} - \dfrac{x}{5} = -17$.

6. $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} = \dfrac{1}{x}$.

7. $\dfrac{1}{10} - \dfrac{1}{x} = \dfrac{1}{30}$.

8. $4x - \dfrac{3x-1}{2} + \dfrac{x}{3} = x + \dfrac{x+3}{6} + \dfrac{5}{3}$. *Ans.* $x = 1$.

9. $x - \dfrac{x-1}{3} - \dfrac{x-2}{5} + \dfrac{x}{2} = 61\dfrac{19}{30}$. *Ans.* $x = 63$.

10. $x - \dfrac{x-2}{3} + \dfrac{x-4}{5} = 7 + \dfrac{x-5}{6}$. *Ans.* $x = 9$.

11. $x + \dfrac{x-7}{5} - \dfrac{x-3}{4} - \dfrac{x-4}{5} = 12\dfrac{9}{10}$. *Ans.* $x = 17$.

12. $x - \dfrac{x-2}{3} - \dfrac{x-4}{5} - \dfrac{x-5}{6} = 20\dfrac{3}{10}$. *Ans.* $x = 60$.

13. $x - \dfrac{x-2}{3} - \dfrac{x-3}{4} - \dfrac{x-4}{5} = 6\dfrac{11}{20}$. *Ans.* $x = 20$.

14. $x - \dfrac{x-13}{5} + \dfrac{x-40}{7} = 21 + \dfrac{3x-2}{35}$. *Ans.* $x = 28\frac{1}{15}$.

15. $x - \dfrac{x-2}{3} - \dfrac{x-4}{5} - \dfrac{x-3}{6} = 19\dfrac{29}{30}$. *Ans.* $x = 60$.

16. $x - \dfrac{x-2}{7} + \dfrac{x-4}{8} - \dfrac{x-7}{2} + \dfrac{x+3}{56} = 10\dfrac{19}{56}$.

17. $\dfrac{3x-4}{2} = \dfrac{x}{2} + \dfrac{x}{4} - \dfrac{1}{2}$. *Ans.* $x = 2$.

18. Solve the equation $7 - x - \frac{2}{9}(2x + 3) = 6 - \frac{5x + 1}{4}$.

*Solution.*

$$7 - x - \frac{2}{9}(2x + 3) = 6 - \frac{5x+1}{4} \qquad (1) = \text{given equation.}$$

$$252 - 36x - 16x - 24 = 216 - 45x - 9 \qquad (2) = (1) \times 36.$$

Transpose & unite and we have $\quad -7x = -21 \qquad (3)$

$$x = 3 \qquad (4)$$

19. $\frac{x-5}{3} + \frac{x-1}{2} + 4(x - 3) = 68$. *Ans.* $x = 17$.

20. $\frac{4}{3}(x + 2) = 3\frac{(x+1)}{2}$. *Ans.* $x = 7$.

21. $10\left(x + \frac{1}{2}\right) - 6x\left(\frac{1}{x} - \frac{1}{3}\right) = 23$. *Ans.* $x = 2$.

22. $3x - \frac{4}{5}\left(\frac{7x-9}{3}\right) = \frac{4}{5}\left(\frac{x+17}{3}\right) + 4$. *Ans.* $x = 7\frac{1}{13}$.

23. $\frac{67x}{8} - \frac{134}{5}(x - 1) = \frac{201x - 268}{15} + \frac{67x}{12}$. *Ans.* $x = 1\frac{13}{67}$.

24. $\frac{9}{7}(x - \frac{1}{2}) - \frac{1}{5}(6 - 9x) = \frac{129}{10}$. *Ans.* $x = 4\frac{7}{9}$.

25. $\frac{x}{6} + 5 = \frac{3}{4}(x + 2)$.

26. $\frac{3}{4}(x + 1) = \frac{5}{6}(x - 1)$.

**101.** When the unknown quantity is found in all the terms, involved to powers no two of which differ by more than unity, the equation may be divided by $x$ involved to the lowest power, and thus reduced to an equation of the first degree.

EXAMPLES.

1. Solve the equation $\frac{x^2}{10} - x = \frac{x^2}{30}$.

Divide by $x$, and we have $\frac{x}{10} - 1 = \frac{x}{30}$. *Ans.* $x = 15$.

2. Solve the equation $\frac{7x}{20} = x^2 - 5x$. *Ans.* $x = 5\frac{7}{20}$.

3. Solve the equation $27\frac{2}{3}x^3 + \frac{498x^3}{15} = \frac{332x^3 - 166x^2}{5}$.

*Ans.* $x = 6$.

4. Solve the equation $\frac{92}{3x^3} + \frac{115}{2x^3} = \frac{46}{x^4 - 10x^3}$. *Ans.* $x = 10\frac{12}{23}$.

5. Solve the equation $\frac{3x}{7} = 135x^0$. *Ans.* $x = 315$.

6. Solve the equation $y + \frac{3y}{1-5y} = \frac{7y^2}{1-5y}$. *Ans.* $y = \frac{1}{3}$.

7. Solve the equation $\frac{-4y}{3-2y} = \frac{8y}{15-y}$. *Ans.* $y = 4\frac{1}{5}$

## LITERAL EQUATIONS.

**102.** Literal Equations need be verified only by introducing some number which each letter may be made to represent into the given equation, together with the corresponding value of $x$.

### EXAMPLES.

1. Solve the equation $\frac{6a^2x}{11} + \frac{3ac}{2} - \frac{3b^2x}{22} = \frac{45ac}{22} + \frac{6bc}{11} + \frac{9b^2x}{22}$.

*Solution.*

$$12a^2x + 33ac - 3b^2x = 45ac + 12bc + 9b^2x.$$
$$12a^2x - 12b^2x = 12ac + 12bc.$$
$$(a^2 - b^2)\,x = (a + b)\,c.$$
$$x = \frac{c}{a-b}.$$

If $a = 5$, $b = 4$ and $c = 20$, then $x = 20$.

Verification:

$$\frac{6.5^2.20}{11} + \frac{3.5.20}{2} - \frac{3.4^2.20}{22} = \frac{45.5.20}{22} + \frac{6.4.20}{11} + \frac{9.4^2.20}{22}.$$

If $a = 7$, $b = 5$ and $c = 30$, then $x = 15$.

2. Solve the equation $\frac{a-x}{3} - \frac{a-2x}{3} + x = b$. *Ans.* $x = \frac{3b}{4}$.

If $a =$ *anything* and $b = 8$, then $x = 6$.

If $b = 12$, $x = 9$. If $b = 20$, $x = 15$, &c.

3. Solve the equation $\frac{1}{a-b} + \frac{1}{a^2-b^2} + \frac{1}{a+b} = \frac{1}{x}$.

*Ans.* $x = \frac{a^2-b^2}{2a+1}$.

If $a = 10$ and $b = 4$, then $x = 4$.

If $a = 7$ and $b = 2$, then $x = 3$.

4. Solve the equation $\frac{1}{2}(a+b)x = a + \frac{ab}{4} - \frac{x}{3}$.

*Ans.* $x = \frac{3a(4+b)}{6(a+b)+4}$.

If $a = 2$ and $b = 2$, then $x = \frac{9}{7}$.

If $a = 1$ and $b = 2$, then $x = \frac{9}{11}$.

5. Solve the equation $2x + \frac{a-x}{2} + \frac{b-x}{3} + \frac{c-x}{4} = a + b + c$.

*Ans.* $x = \frac{6a+8b+9c}{11}$.

If $a = 11$, $b = 11$ and $c = 11$, then $x = 23$.

If $a = 1$, $b = 2$ and $c = 3$, then $x = 4\frac{5}{11}$.

6. Solve the equation $x - \frac{x-a}{2} - \frac{x-b}{3} = a - \frac{b}{6}$.

*Ans.* $x = 3(a-b)$.

If $a = 8$ and $b = 5$, then $x = 9$.

If $a = 3$ and $b = 2$, then $x = 3$.

7. Solve the equation

$$x + \frac{x-a}{3} + \frac{x-b}{5} + \frac{x-c}{6} = \frac{41a+45b+46c}{30}.$$

*Ans.* $x = a + b + c$

If $a = 1$, $b = 2$ and $c = 3$, then $x = 6$.

8. Solve the equation

$$x + \frac{x-a^4}{7} - \frac{x-a^2c^2}{5} - \frac{x-c^4}{35} = -\frac{4a^4}{35} + \frac{a^2c^2}{7} + \frac{c}{17\frac{1}{2}}.$$

*Ans.* $x = \frac{(a^2-c^2)^2}{32}$.

If $a = 3$ and $c = 1$, then $x = 2$, &c.

9. Solve the equation $\frac{x-a^2}{2}+\frac{x-ab}{3}-\frac{7a^2}{12}-\frac{11ab}{6}=\frac{5b^2}{6}-\frac{x-b^2}{4}$. *Ans.* $x=(a+b)^2$.

If $a=1$, $b=2$, then $x=9$.

10. Solve the equation $\frac{1+x}{1-x}=1+\frac{1}{a}$. *Ans.* $x=\frac{1}{2a+1}$.

If $a=1, 2, 3, 4$, &c., then $x=\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$, &c.

11. Solve the equation $\frac{1+a}{1-a}=1+\frac{1}{x}$. *Ans.* $x=\frac{1-a}{2a}$.

If $a=\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$, &c., then $x=1, 2, 3, 4$, &c.

## 103. PROBLEMS.

(1.) A *Problem* is a question proposed for solution.

(2.) Any algebraic equation can be considered a *statement*, in algebraic language, of the conditions of some problem.

(3.) Any algebraic problem, if properly expressed, can be converted into one or more algebraic equations, called the *equations of the problem.*

(4.) The problem is solved by solving the equations. (*Vide* **100.**)

(5.) The statement of a problem may generally be effected by considering $x$ as the answer sought, and *indicating* the operations that would *actually be performed*, if the *value* of $x$ were known, in the verification of the problem.

### EXAMPLES.

1. One-third of a certain number is 7. What is the number?

Let $x$ = the number.

Then $\frac{x}{3}=7$, (1.)

and $x=21$. (*Vide* **97**, ex. 1.) (2.)

Now if we perform the operation on 21 which is *indicated* on $x$ in equation (1), the result will be verified, $\frac{21}{3}=7$.

2. If Charles had twice as many marbles as he now has, and also three times as many, he would have as many as John and William together, the former of whom has 30, and the latter half as many. How many has Charles?

$$\text{Let } x = \text{his marbles.}$$

$$\text{Then,} \quad 2x + 3x = 30 + 15.$$

$$\text{Whence,} \quad x = 9. \qquad (\textit{Vide } \mathbf{96}, \text{ ex. } 14.)$$

Charles, therefore, has 9 marbles, for $2 \times 9 + 3 \times 9 = 30 + 15$.

3. What number is that whose half and third added together make 10? (*Vide* **98**, ex. 14.) *Ans.* 12.

4. *Three-fourths of a number added to two-fifths of it make 23. What is the number? (*Vide* **98**, ex. 16.) *Ans.* 20.

5. *If a number is added to its half and third, the sum will be 11. What is the number? (*Vide* **98**, 17.) *Ans.* 6.

6. *If the fourth of a number be subtracted from the sum of its half and third, the result will be 7. What is the number? (*Vide* **98**, 18.) *Ans.* 12.

7. *If one forty-fifth of all the sheep I have be subtracted from the sum of two-fifths and four-ninths of them, the result will be 37. How many sheep have I? (*Vide* **98**, 19, also **103**, 2.) *Ans.* 45.

8. If from nine times a certain number 5 be subtracted, the remainder will be 40. What is the number? (*Vide* **99**, ex. 1.) *Ans.* 5.

9. *If the fourth of a certain number increased by 1 is subtracted from half of the same number, the remainder will be $\frac{9}{2}$. What is the number? (*Vide* **100**, ex. 1.)

10. *Four-thirds of a number increased by 2 is the same as three halves of the same number increased by 1. What is the number? (*Vide* **100**, ex. 20.) *Ans.* 7.

11. *If 5 be added to the sixth of a number it will make the same thing as three-fourths of the number increased by 2. What is the number? (*Vide* **100**, ex. 25.) *Ans.* 6.

12. *If from a number its half, its third, and three more be subtracted, the remainder will be 1. What is the number? *Ans.* 24.

13. *The difference between the fifth and sixth of a number is 4. What is the number? $\frac{x}{5} - \frac{x}{6} = 4.$ *Ans.* 120.

14. †If from a number we take 2, and divide the remainder by 11, the quotient will be 6. $\frac{x-2}{11} = 6.$ *Ans.* 68.

15. *The sum of two-thirds and three-fourths of a number is 68. What is the number? *Ans.* 48.

16. †If 4 be added to a number, one-third of the sum will be 5. What is the number. *Ans.* 11.

17. †If 3 be subtracted from a number, two-thirds of the remainder will be 16. What is the number? *Ans.* 27.

18. *In one flock a man has one-fourth of all his sheep, in another one-sixth, in another one-eighth, in another one-twelfth, and in another 450 sheep. These five flocks are all he has. How many sheep has he, and how many in each flock?

*Ans.* $1200 = \overset{(1.)}{300} + \overset{(2.)}{200} + \overset{(3.)}{150} + \overset{(4.)}{100} + \overset{(5.)}{450}.$

19. A certain number added to ten times itself gives 132. What is the number? *Ans.* 12.

20. A gold watch is worth ten times as much as a silver watch, and both together are worth \$132. What is each watch worth? *Ans.* \$120 and \$12.

21. A man paid \$74 for a sheep, a cow and an ox. The cow was valued at 12 sheep, and the ox at 2 cows. What was the price of each? *Ans.* \$2, \$24, \$48.

22. A key winds both a gold and a silver watch. The silver watch is worth twelve times the key, and the gold watch twenty-five times the key. What is the value of each, if all are worth \$342? *Ans.* key \$9, silver watch \$108, gold watch \$225.

23. A man paid \$460 for 20 sheep, 5 cows and a *yoke* of oxen. A cow was valued at 8 sheep, and an ox at 2 cows. What was the price, per head, of each? *Ans.* \$5, \$40 and \$80.

24. *Three men and two boys work together. The men get a quarter of a dollar per day, the boys one-fifth of a dollar. How many days must they work to receive 23 dollars?
(*Vide* **98**, ex. 16, also **103**, (2) and ex. 4.) *Ans.* 20 days.

The pupil will perceive that any equation may be that of an endless variety of problems, but that these problems are only different methods of expressing the *same conditions*, as the uniform *statement* proves.

25. A starts from a given point, and travels at the rate of *one* mile per hour. After an absence of 12 hours, B starts after him on the same route, at the rate of *twelve* miles per hour. How long before A will be overtaken, and how far will B have traveled?

*Solution.*


Let $M\ N$ represent the road traveled over.

Let $x =$ the number of hours required.

Since A goes *one* mile per hour, in 12 hours he will go 12 miles $= M\ P$.

Since A goes *one* mile per hour, in $x$ hours he will go $x$ miles $= P\ N$.

Since B goes 12 miles per hour, in $x$ hours he will go $12x$ miles $= M\ N$.

Now $\quad MN = MP + PN$

That is $\quad 12x = 12 + x$

Whence $\quad x = 1\frac{1}{11} = 1$ hour $5\frac{5}{11}$ minutes.

Now B's distance being $12x$, he will have traveled $13\frac{1}{11}$ miles.

26. The hour and minute hands of a clock are together at 12. When will they be together again? *Ans.* 1 hour, $5\frac{5}{11}$ min.

27. Two men start from the same point, and travel in the *same* direction; the first steps twice as far as the second, but the second makes five steps while the first makes one. At the end of a certain time they are 300 feet apart. How far has each traveled? $2\frac{1}{2}x = 300 + x$. *Ans.* 1st 200, 2nd 500 feet.

28. Two men start from the same point, and travel in *opposite* directions; the first steps, each time, two-thirds the distance of the second; but the second makes only 4 steps while the first makes 7. At the end of a certain time they are 520 feet apart. How far has each traveled? *Ans.* 1st 280, 2nd 240.

29. A cistern has three pipes. The first will fill it in 2 ($1\frac{1}{3}$) hours, the second in 3 ($3\frac{1}{3}$) hours, the third in 4 (5) hours. In what time will the cistern be filled when the three pipes are running together.

*Solution.*

Let $x =$ the time; then,

$\frac{1}{2} =$ the part the 1*st* will fill in one hour.

$\frac{1}{3} =$ " " 2*nd* " " " "

$\frac{1}{4} =$ " " 3*rd* " " " "

$\frac{1}{x} =$ the part all will fill in one hour.

Hence, (axiom 1) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{1}{x}$, whence $x = \frac{12}{13}$ of an hour.

(*Vide* **100**, ex. 6.)

30. Solve the above problem using the numbers in the ( ).

*Ans.* 48 minutes.

31. A cistern has three pipes, two at the top and one at the bottom. One of the top pipes would fill it in 5 hours, the other

in 6; but the pipe at the bottom empties it in $8\frac{4}{7}$ hours. In what time will the cistern be filled when the pipes are running together? *Ans.* 4 hours.

32. A can do a piece of work in 3 ($2\frac{2}{3}$) days, B in 5 (2) days, and C in $7\frac{1}{2}$ (8) days. In what time can they all do it by working together? *Ans.* $1\frac{1}{2}$ days.

33. A and B can do a piece of work in 5 ($\frac{6}{11}$) days. A can do it alone in 7 ($7\frac{1}{3}$) days. In what time can B do it alone? *Ans.* $17\frac{1}{2}$ days.

34. A and B can do a piece of work in 5 days; A and C in $6\frac{4}{11}$ days; B and C in 7 days. In what time would all do it by working together? $\frac{1}{5}+\frac{1}{7}+\frac{11}{70}=\frac{2}{x}$. *Ans.* 4 days.

35. A man and his wife could drink a cask of beer in 10 days. In the absence of the man it lasted his wife 30 days. How long would the man be occupied in drinking it? *Ans.* 15 days.

36. A, B and C could do a piece of work in $\frac{6}{11}$ days; A, B and D in $\frac{4}{7}$ days; A, C and D in $\frac{12}{19}$ days; B, C and D in $\frac{12}{13}$ days. In what time could they all do the work, and in what time could each man do it alone? *Ans.* All in $\frac{12}{25}$ days; A in 1; B in 2; C in 3; and D in 4 days.

37. Divide 55 (80) into two parts, so that the less (greater) part divided by the difference (sum) of the parts shall be 2 ($\frac{3}{4}$). *Ans.* 33 and 22.

38. Four places are situated in the order of the four letters A, B, C and D. The distance from A to D is 134 miles. The distance from C to D is $\frac{2}{3}$ the distance from A to B, and $\frac{1}{4}$ the distance from A to B added to half the distance from C to D is three times the distance from B to C. What are the distances?

39. A person went to a tavern, where he spent 5 shillings, and then borrowed twice as much as he had left. He does the

same at a second and a third tavern; but on spending 21 shillings at a fourth tavern he had nothing left. How much had he at first? *Ans.* 8 shillings.

40. A boy had a number of marbles. He laid aside 2, and then won in play as many as he had left. He then laid aside 3 more, and again won as many as were left. He now adds 4 more to the reserved pile, and wins, as before, as many as he has left. Then counting his marbles he finds 13. How many did he begin with? *Ans.* 5.

41. Two boys, Charles and John, play marbles. First game, Charles wins 4 marbles. Second game, John wins 12. Charles again wins 4 in the third game, and John wins 6 in the fourth and last game. John now has three times as many marbles as Charles, although each had the same number when the play commenced. How many marbles had each at first? *Ans.* 20.

42. A commenced trade, and at the end of the third year found his original stock tripled. Had his gains been $1000 per year more than they actually were, he would have doubled his stock each year. What was his original stock? *Ans.* $1400.

43. Divide the number 20 into two parts, so that the product of the parts shall be 5 times the greater part.

Let $x =$ the greater part, and $20 - x$ the less.

Then $20x - x^2 = 5x$, whence 15 and 5 are the numbers.

44. Divide the number 40 into two parts, so that the product of the parts may be 35 times the smaller part.

45. A boatman rows 14 miles an hour with the tide. *Against* a tide two-thirds as strong he rows only 4 miles an hour. What is the velocity of the tide in each case? *Ans.* 6 and 4 miles.

46. Three persons, A, B and C, were seen traveling in the same direction. At first A and B were together, and C 12 miles

in advance of them. A goes 7, B 10, and C 5 miles per hour. In what time will B be half way between A and C? How long before C will be midway between A and B? How long since A was midway between B and C?

*Ans.* respectively $1h.$ $30m.$, $3h.$ $25\frac{5}{7}m.$, and $12h.$

**104.** (1.) It is often much more convenient to represent the unknown quantity by such an expression as will avoid the introduction of fractions into the equation of the problem. It is in fact a much better exercise to solve a single problem in several different ways than to be engaged on as many different problems. The *shortest* method of solution should always be found out, as it leads to the clearest insight into the problem.

EXAMPLES.

1. What number is that whose half and third added together make 10? (*Vide* **103**, ex. 3.)

Let $6x$ = the number.

Then $3x + 2x = 10$, whence $x = 2$ and $6x = 12$.

In the same way solve those marked * in the preceding section; *i. e.*, let the unknown quantity be represented by the least common multiple of the denominators of the fractions in the problems.

2. The rent of an estate is this year 5 per cent, *i. e.* $\frac{1}{20}$, greater than it was last year. This year it is 840 dollars. What was it last year? Let $20x$ = the rent last year. *Ans.* \$800.

3. If from a number we take 2, and divide the remainder by 11, the quotient will be 6. What is the number?

Let $11x + 2$ = the number.

Then $x = 6$, and $11x + 2 = 68$. (*Vide* **103**, ex. 14.)

In a similar manner solve those marked † of section **103.**

4. What number is that from which if 5 be subtracted $\frac{2}{3}$ of the remainder will be 40? $3x + 5$. *Ans.* 65.

5. What number is that to which if 7 be added $\frac{2}{3}$ of the sum will be 18? $3x - 7$. *Ans.* 20.

6. A teacher spent $\frac{2}{5}$ of his salary for board and lodging, $\frac{1}{3}$ of the remainder for clothes, $\frac{1}{6}$ of what remained for books, and saved \$120 per annum. What was his salary? $15x =$ salary. *Ans.* \$360.

7. In a mixture of wine $\frac{1}{2}$ the whole, plus 25 gallons, was wine; $\frac{1}{3}$ the whole, minus 5 gallons, was water. What was the quantity of each in the mixture?

Let $6x =$ the whole, then $3x + 25 + 2x - 5 = 6x$.

8. One-half of a certain number is the same as $\frac{1}{3}$ another number. But if 5 is added to the first and 10 to the second, then $\frac{1}{5}$ of the first is the same as $\frac{1}{8}$ of the second. What are the numbers? $2x$ and $3x$. *Ans.* 20 and 30.

9. Divide 90 into four parts so that if the first be diminished by 2, the second increased by 2, the third divided by 2, and the fourth multiplied by 2, the results will be equal.

Let $2x =$ the quantity to which they are to be equal.

$$\text{Then } \overset{\text{1st part.}}{\overline{2x+2}} + \overset{\text{2d part.}}{\overline{2x-2}} + \overset{\text{3d part.}}{\overline{4x}} + \overset{\text{4th part.}}{\overline{x}} = 90, \text{ whence } x = 10.$$

And 22, 18, 40, 10 are the parts.

10. Divide the number 151 into 5 parts so that twice the 1*st*, one-half the 2*d*, one-third the 3*d*, one-fifth the 4*th*, and three and one-half times the 5*th* shall be equal.

Let $14x =$ the quantity to which they are to be equal.

11. A person supported himself 3 years for \$50 a-year. At the end of each year he added to that part of his stock which was not thus expended a sum equal to $\frac{1}{3}$ of this part. At the

end of the third year his original stock was doubled. What was the amount of stock at first?

Let $27x + 200 =$ the original stock.

Then $27x + 150 =$ the original stock less \$50.

And $9x + 50 =$ one-third this remainder.

$36x + 200 =$ stock at the close of first year.

In the same way $48x + 200 =$ " " " second "

And $64x + 200 =$ " " " third "

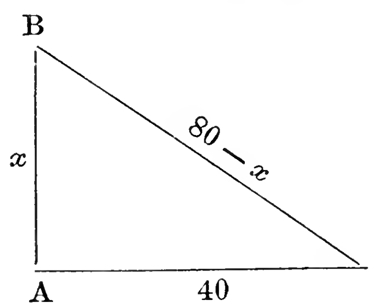
Therefore $64x + 200 = 54x + 400$ by the question.

Whence $x = 20$

And $27x + 200 =$ \$740 the original stock.

12. From a certain sum of money I took one-third part, and put in its place \$50. From this sum I took one-tenth part, and soon replaced it with \$37, when the sum amounted to \$100. What was there at first? $15x - 75$.

13. A tree 80 feet high was broken by the wind in such a manner that the top reached the ground just 40 feet from the bottom of the tree. How high up was the tree broken?




Let $x =$ the distance from the bottom, and $80 - x$ the part broken off.

Then $(80 - x)^2 = x^2 + 40^2$ Euclid, Bk. I, 47.

Or $6400 - 160x + x^2 = x^2 + 1600$

$x = 30 =$ A B. $80 - x = 50 =$ B C.

14. Two trees 80 and 60 feet high stand on the same horizontal plane, 100 feet apart. Where must a person stand to be equally distant from the top of each? (*Vide* **121**, ex. 4.)

*Ans.* 64 feet from the shorter tree, or 36 feet from the taller.

## EQUATIONS OF THE FIRST DEGREE INVOLVING TWO UNKNOWN QUANTITIES.

**105.** *Simultaneous equations* are those in which the values of the unknown quantities are the same in both. Thus,

$$x + y = 30 \text{ and } x - y = 6$$

are simultaneous equations, because either of them can be verified when $x = 18$ and $y = 12$.

**106.** Simultaneous equations are *independent* of each other when one is not a mere transformation of the other, or when one equation is not a result of the combination of two or more equations.

Thus, $x + y = 30$ and $x - y = 6$ are independent simultaneous equations; but, $x + y = 30$ and $3x = 90 - 3y$ are dependent, since the first may be easily obtained from the second.

Also, $3x + 2y + z = 10$, $x + y + z = 6$ and $x + 2y + 3z = 14$ are dependent, since the second is one-fourth the sum of the other two.

## ELIMINATION.

**107.** *Elimination* is the operation of combining two equations in such a manner as to cause one of the unknown quantities to disappear in a new equation.

There are three principal methods of elimination, by *addition or subtraction*, by *substitution*, and by *comparison*.

## ELIMINATION BY ADDITION OR SUBTRACTION.

**108.** 1. Resume the equations,

$$x + y = 30 \qquad (1)$$

and

$$x - y = 6 \qquad (2)$$

By axiom 2, we may add these equations together. Doing so we have $2x = 36$ (3) $= (1) + (2)$. *Vide* **34**, ex. 36.
Whence $x = 18$ (4) $= (3) \div 2$.

By axiom 3, we may subtract (2) from (1). Doing so we have $2y = 24$ (5) $= (1) - (2)$. *Vide* **45.**
Whence $y = 12$ (6) $= (5) \div 2$.

By putting the values of $x$ and $y$ in place of these letters in (1) and (2), we have $18 + 12 = 30$
and $18 - 12 = 6$ *Vide* **105.**

In the same way find the values of $x$ and $y$ in the following sets of equations.

| | | | |
|---|---|---|---|
| $x + y = 10$ | $x + y = 12$ | $x + y = 20$ | $x + y = 25\frac{1}{2}$ |
| $x - y = 4$ | $x - y = 8$ | $x - y = 15$ | $x - y = 3\frac{1}{2}$ |

2. Again; take the equations,

$$3x + 2y = 22 \qquad (1)$$

and

$$2x + 3y = 23 \qquad (2)$$

By axiom 4, we may multiply (1) by 2 and (2) by 3.
This gives $6x + 4y = 44$ (3) $= (1) \times 2$
and $6x + 9y = 69$ (4) $= (2) \times 3$

By axiom 3, subtract (3) from (4), and we have,

$$5y = 25 \qquad (5) \qquad = (4) - (3)$$

Whence $y = 5$ (6) $= (5) \div 5$

By axiom 4, we may multiply (1) by 3 and (2) by 2.
This gives $9x + 6y = 66$ (7) $= (1) \times 3$
and $4x + 6y = 46$ (8) $= (2) \times 2$

By axiom 3, subtract (8) from (7), and we have,

$$5x = 20 \qquad (9) \qquad = (7) - (8)$$

Whence $$x = 4 \qquad (10) \qquad = (9) \div 5$$

By putting the values of $x$ and $y$ in place of these letters in (1) and (2), we have $3.4 + 2.5 = 22$

and $$2.4 + 3.5 = 23$$

which proves that the values of $x$ and $y$ are correct.

We multiplied equation (1) by 2, and equation (2) by 3, simply *to make the coëfficients of* x *in these equations alike*, and because the *signs* before the *like* coëfficients of equations (3) and (4) are *alike;* by *subtracting*, $x$ disappears in the resulting equation (5), where there is only the letter $y$, whose value in (6) is obtained in the manner heretofore explained.

We now multiply equation (1) by 3, and equation (2) by 2, *to make the coëfficients of* y *alike*, which leads to the value of $x$, in the very same way as before.

By this process we have *eliminated* $x$ and found the value of $y$. We then *eliminated* $y$ and found the value of $x$.

3. Again; take the equations,

| | | | |
|---|---|---|---|
| | $5x + 3y = 13$ | (1) | |
| and | $3x - 7y = -1$ | (2) | |
| By ax. 4, | $35x + 21y = 91$ | (3) | $= (1) \times 7$ |
| By ax. 4, | $9x - 21y = -3$ | (4) | $= (2) \times 3$ |
| By ax. 2, | $44x = 88$ | (5) | $= (3) + (4)$ |
| By ax. 5, | $x = 2$ | (6) | $= (5) \div 44$ |
| By ax. 4, | $15x + 9y = 39$ | (7) | $= (1) \times 3$ |
| By ax. 4, | $15x - 35y = -5$ | (8) | $= (2) \times 5$ |
| By ax. 3, | $44y = 44$ | (9) | $= (7) - (8)$ |
| By ax. 5, | $y = 1$ | (10) | $= (9) \div 44$ |

In this example, after making the *coëfficients* of $y$ *alike*, because the *signs* of these coëfficients are *unlike*, we *add* equations (3)

and (4), and $y$ disappears. In other particulars the steps are the same as in the previous example.

4. Take the equations,

$$\frac{x}{2} + \frac{y}{3} = 3\tfrac{2}{3} \quad (1)$$

and $$\frac{x}{3} + \frac{y}{2} = 3\tfrac{5}{6} \quad (2)$$

By ax. 4, $3x + 2y = 22$ (3) $= (1) \times 6$ (*Vide* **98.**)

By ax. 4, $2x + 3y = 23$ (4) $= (2) \times 6$ (*Vide* **98.**)

Equations (3) and (4) are the same as (1) and (2) of ex. 2. They should be treated in like manner.

5. Take the equations,

$$\frac{x}{2} + \frac{y}{10} = 1\tfrac{3}{10} - \frac{y}{5} \quad (1)$$

and $$\frac{x}{7} - \frac{y}{2} = -\frac{1+x}{14} \quad (2)$$

By ax. 4, $5x + y = 13 - 2y$ (3) $= (1) \times 10$

By ax. 4, $2x - 7y = -1 - x$ (4) $= (2) \times 14$

*Vide* **99**, $5x + 3y = 13$ (5) $= (3)$ transposed.

*Vide* **99**, $3x - 7y = -1$ (6) $= (4)$ transposed.

The equations (5) and (6) are (1) and (2) of ex. 3, and should be treated in the same manner. Therefore,

**109.** Having two equations with two unknown quantities, to find their values by the method of elimination, by addition or subtraction:

(1.) If necessary, *clear the equations of fractions.*

(2.) In each equation collect all the terms involving $x$ into one term, and write this term *first* in the *first member* of a new equation, *prefixing the correct sign.*

(3.) In each equation unite all the terms involving $y$ in one

term, and write this term *second* in the *first member* of the new equation, *prefixing the correct sign.*

(4.) In each equation collect the known quantities into one term, and write this term in the *second member* of the corresponding new equation, *prefixing the correct sign.*

(5.) Find the *least common multiple* of the coëfficients of the letter you wish to *eliminate*, and multiply the equations respectively by the quotient of this multiple divided by the *coëfficient of the same letter* in the equation to be multiplied. The resulting coëfficients of this letter will be the same.

(6.) If then the *signs* of these coëfficients are *alike*, *subtract one equation from the other*; if the *signs* of these coëfficients are *unlike*, *add one equation to the other*. (*Vide* **40** and **34**.)

(7.) From the resulting equation find one of the unknown quantities, and by repeating the steps 5 and 6, find the other.

These directions should be followed till the process is perfectly at the command of the pupil. The notation on the right should be invariably demanded, since it leads to habits of accuracy.

## EXAMPLES.

1. Find $x$ and $y$ in the equations,

$$21 - 6y = \frac{x+8}{4} \qquad (1)$$

and

$$23 - 5x = \frac{y+6}{3} \qquad (2)$$

| | | |
|---|---|---|
| $84 - 24y = x + 8$ | (3) | $= (1) \times 4$ |
| $69 - 15x = y + 6$ | (4) | $= (2) \times 3$ |
| $x + 24y = 76$ | (5) | $= (3)$ transposed |
| $15x + y = 63$ | (6) | $= (4)$ transposed |
| $15x + 360y = 1140$ | (7) | $= (5) \times 15$ |
| $359y = 1077$ | (8) | $= (7) - (6)$ |
| $y = 3$ | (9) | $= (8) \div 359$ |

$$360x + 24y = 1512 \qquad (10) \qquad = (6) \times 24$$
$$359x = 1436 \qquad (11) \qquad = (10) - (5)$$
$$x = 4 \qquad (12) \qquad = (11) \div 359$$

2. $\left.\begin{aligned} 2x + 7y &= 38 \\ 3x - 5y &= -5 \end{aligned}\right\}$

3. $\left.\begin{aligned} 7x - 3y &= 10 \\ 8x + 2y &= 120 \end{aligned}\right\}$

4. $\left.\begin{aligned} 5x - 4y &= 7 \\ 10x + 5y &= 40 \end{aligned}\right\}$

5. $\left.\begin{aligned} 7x - 9y &= -2 \\ 8x + 21y &= 29 \end{aligned}\right\}$

6. $\left.\begin{aligned} 12x - 7y &= 12 \\ 11x - 3y &= 11 \end{aligned}\right\}$

7. $\left.\begin{aligned} 4x + 3y &= 65 \\ 5x - 2y &= 41 \end{aligned}\right\}$

8. $\left.\begin{aligned} 3x + 5y &= 15 \\ 4x + y &= 3 \end{aligned}\right\}$

9. $\left.\begin{aligned} 7x + 5y &= 2 \\ 14x - 10y &= 0 \end{aligned}\right\}$

10. $\left.\begin{aligned} \frac{x}{2} + \frac{y}{3} &= 9 \\ \frac{2x}{3} - \frac{4y}{5} &= -2\frac{14}{15} \end{aligned}\right\}$

11. $\left.\begin{aligned} \frac{2x}{7} + \frac{3y}{5} &= 10 \\ 2x - \frac{5y}{13} &= 24\tfrac{2}{3} \end{aligned}\right\}$

12. $\left.\begin{aligned} \frac{3x}{13} + \frac{2y}{11} &= 10 \\ \frac{3x}{52} + \frac{y}{44} &= 2 \end{aligned}\right\}$

13. $\left.\begin{aligned} \frac{x}{7} - \frac{7y}{10} &= -20 \\ \frac{x}{4} + 3y &= 134 \end{aligned}\right\}$

14. $\left.\begin{aligned} \frac{x}{2} &= \frac{y}{4} + 20 \\ \frac{x+y}{5} + \frac{x}{3} &= \frac{2y-x}{4} + 35 \end{aligned}\right\}$

15. $\left.\begin{aligned} \frac{y}{7} - \frac{x}{6} &= -3 \\ x + \frac{y}{5} &= 17\tfrac{4}{5} \end{aligned}\right\}$

16. $\left.\begin{aligned} \frac{y}{7} - \frac{3x}{2} &= -8 \\ 5x - \frac{y}{7} &= 29 \end{aligned}\right\}$

17. $\left.\begin{aligned} \frac{2x-3y}{5} &= x - 2\tfrac{2}{5} \\ x - \frac{y-1}{2} &= 0 \end{aligned}\right\}$

18. $\left.\begin{aligned} \frac{x+y}{2} + \frac{x-y}{2} + \frac{x}{3} &= \frac{3y}{4} \\ \frac{x}{3} + \frac{y}{4} - \frac{2x+3y}{5} &= \frac{4x}{7} \end{aligned}\right\}$

19. $\left.\begin{aligned} \frac{x}{2} + \frac{y}{3} &= 22 \\ \frac{x}{3} + \frac{y}{4} &= 16 \end{aligned}\right\}$

20. $\left.\begin{array}{l}\frac{1}{x}+\frac{1}{y}=\frac{5}{6}\\ \frac{1}{x}-\frac{1}{y}=\frac{1}{6}\end{array}\right\}$

21. $\left.\begin{array}{l}\frac{2x}{3}+\frac{3y}{4}=26\\ \frac{3x}{4}-\frac{2y}{3}=-7\end{array}\right\}$

22. $\left.\begin{array}{l}\frac{4x}{7}+\frac{9y}{14}=20\\ \frac{9x}{14}-\frac{4y}{7}=-13\frac{3}{4}\end{array}\right\}$

23.* $\left.\begin{array}{l}\frac{x}{7}+7y=99\\ 7x+\frac{y}{7}=51\end{array}\right\}$

24.* $\left.\begin{array}{l}\frac{3x}{5}+\frac{5y}{3}=59\\ \frac{5x}{3}+\frac{3y}{5}=43\end{array}\right\}$

25.* $\left.\begin{array}{l}x+\frac{y}{49}=51\\ y+\frac{x}{49}=99\end{array}\right\}$

26.* $\left.\begin{array}{l}x+\frac{y}{15}=17\\ y+\frac{x}{15}=31\end{array}\right\}$

27.* $\left.\begin{array}{l}\frac{3x}{5}+\frac{2y}{7}=22\\ \frac{2x}{7}+\frac{3y}{5}=16\frac{34}{35}\end{array}\right\}$

28.* $\left.\begin{array}{l}\frac{4x}{5}-\frac{3y}{7}=-2\\ \frac{3x}{7}-\frac{4y}{5}=-41\end{array}\right\}$

29.* $\left.\begin{array}{l}4x+3y=3\\ 3x+4y=4\end{array}\right\}$

30.* $\left.\begin{array}{l}\frac{4x}{7}+\frac{3y}{8}=37\\ \frac{3x}{8}+\frac{4y}{7}=42\frac{1}{2}\end{array}\right\}$

31.* $\left.\begin{array}{l}4x+3y=7\\ 3x+4y=7\end{array}\right\}$

## ELIMINATION BY SUBSTITUTION.

**110.** 1. Resume the equations,

$$x+y=30 \qquad (1)$$

and

$$x-y=\ 6 \qquad (2)$$

Transpose $y$ in equation (2), and we have

$$x=6+y \qquad (3)$$

*Substitute* this *value of* $x$ for $x$ in equation (1), and we have

$$6 + y + y = 30 \qquad (4)$$

whence $$y = 12 \qquad (5)$$

*Substitute* this *value of* $y$ for $y$ in equation (1), and we have

$$x + 12 = 30 \qquad (6)$$

whence $$x = 18 \qquad (7)$$

**2.** Take the equations, $$\frac{x}{3} + \frac{y}{5} = 6\tfrac{1}{3} \qquad (1)$$

and $$\frac{x}{2} + \frac{y}{3} = 10 \qquad (2)$$

Clear (1) of fractions $5x + 3y = 95 \qquad (3) \qquad = (1) \times 15$

Clear (2) of fractions $3x + 2y = 60 \qquad (4) \qquad = (2) \times 6$

Find $x$ in (4) $$x = \frac{60 - 2y}{3} \qquad (5)$$

*Substitute* this for $x$ in (3) $$\frac{300 - 10y}{3} + 3y = 95 \qquad (6)$$

whence $$y = 15 \qquad (7)$$

and $$x = \frac{60 - 30}{3} = 10 \quad (8) \quad = (5) \text{ in which } 2y = 30$$

It is evident that these steps may be taken on any two equations, hence,

**111.** Having two equations with two unknown quantities, to find their values by the method of *elimination by substitution.*

(1.) If necessary, *clear the equations of fractions.*

(2.) Find, in either of the equations, the value of *one* of the unknown quantities in terms of the *other*, and *substitute* this value for the *same unknown quantity* in the *other equation.*

(3.) From the equation thus formed, find the value of the letter involved.

(4.) Substitute this last value for the letter to which it is equal in *any* equation *except that from which it was obtained*, and find the value of the other letter.

EXAMPLES.

3. Find $x$ and $y$ in the equations,

$$\frac{5x}{11} - \frac{x-y}{4} = 4 \qquad (1)$$

and $$2x + 3y = 43 \qquad (2)$$

$$20x - 11x + 11y = 176 \qquad (3) \qquad = (1) \times 44$$

$$9x + 11y = 176 \qquad (4) \qquad = (3) \text{ reduced}$$

$$x = \frac{43 - 3y}{2} \qquad (5) \qquad = (2) \textit{ vide } \text{above, } 2$$

$$\frac{387 - 27y}{2} + 11y = 176 \qquad (6) \qquad = (4) \textit{ vide } \text{above, } 2$$

whence $$y = 7 \qquad (7) \qquad \textit{vide } \text{above, } 3$$

and $$x = \frac{43 - 21}{2} = 11 \qquad (8) \qquad \textit{vide } \text{above, } 4$$

4. $$\left.\begin{array}{l} 3x + 4y = 18 \\ 2x - \phantom{4}y = \phantom{1}1 \end{array}\right\}$$

5.* $$\left.\begin{array}{l} 4x + 3y = 16 \\ 3x + 4y = 19 \end{array}\right\}$$

6. $$\left.\begin{array}{l} \frac{y}{8} - \frac{x}{4} = -1 \\ \frac{x}{4} - \frac{y}{5\frac{1}{3}} = \phantom{-}0 \end{array}\right\}$$

7. $$\left.\begin{array}{l} x + \frac{x}{2} - 10 = \frac{y}{5} \\ \frac{x}{8} + \frac{y}{5} = 3 \end{array}\right\}$$

8. $$\left.\begin{array}{l} 1 - \frac{y-x}{6} = y - 20\frac{2}{3} \\ \frac{y}{5} = \frac{x}{5} + 2 \end{array}\right\}$$

9. $$\left.\begin{array}{l} \frac{2x - 3y}{5} = x - 2\frac{2}{5} \\ x - \frac{y-1}{2} = 0 \end{array}\right\}$$

10. $$\left.\begin{array}{l} \frac{x}{5} + \frac{y}{4} = 2 \\ \frac{x}{5} - \frac{y}{2} = -1 \end{array}\right\}$$

11. $$\left.\begin{array}{l} \frac{x-2}{5} - \frac{10-x}{3} = \frac{y-10}{4} \\ \frac{2y+4}{3} - \frac{2x+y}{8} = \frac{x+13}{4} \end{array}\right\}$$

12. $$\left.\begin{array}{l} \frac{3x+4y}{5} = \frac{40-x}{4} \\ 2x - \frac{2y}{3} = \frac{84-y}{6} \end{array}\right\}$$

13.* $\left.\begin{array}{l}\dfrac{x}{2}+\dfrac{y}{3}=7\\[2ex]\dfrac{x}{3}+\dfrac{y}{2}=8\end{array}\right\}$

14. $\left.\begin{array}{l}\dfrac{x+y}{10}+\dfrac{x-y}{2}=0\\[2ex]\dfrac{x+y}{5}+\dfrac{x-y}{2}=1\end{array}\right\}$

15. $\left.\begin{array}{l}\dfrac{2x-y}{4}-\dfrac{3}{2}=\dfrac{3y-4x-8}{4}\\[2ex]\dfrac{x+y}{3}=2\frac{2}{3}\end{array}\right\}$

16. Find $x$ and $y$ in the equations,

$$\frac{2}{x}+\frac{2}{y}=10 \qquad (1)$$

$$\frac{1}{x}-\frac{3}{y}=-7 \qquad (2)$$

$$x+y=5xy \qquad (3) \qquad =(1)\times xy \text{ and } \div \text{ by } 2$$

$$y-3x=-7xy \qquad (4) \qquad =(2)\times xy$$

$$x-5xy=-y \qquad (5) \qquad =(3) \text{ transposed}$$

$$(1-5y)\,x=-y \qquad (6) \qquad =(5) \text{ factored}$$

$$x=\frac{-y}{1-5y} \qquad (7) \qquad =(6) \textit{ vide} \text{ above, } 2$$

$$y+\frac{3y}{1-5y}=\frac{7y^2}{1-5y} \qquad (8) \qquad =(4) \textit{ vide} \text{ above, } 2$$

whence $y=\frac{1}{3}$, $x=\frac{1}{2}$ *vide* **101**, ex. 6.

17.† $\left.\begin{array}{l}\dfrac{4}{x}+\dfrac{3}{y}=3\\[2ex]\dfrac{3}{x}-\dfrac{5}{y}=0\end{array}\right\}\begin{array}{l}x=1\frac{11}{15}\\[2ex]y=3\frac{2}{9}\end{array}$

19.† $\left.\begin{array}{l}\dfrac{4}{x}+\dfrac{3}{y}=3\\[2ex]\dfrac{5}{x}-\dfrac{6}{y}=\frac{1}{2}\end{array}\right\}$

18.† $\left.\begin{array}{l}\dfrac{5}{x}-\dfrac{6}{y}=\frac{1}{2}\\[2ex]\dfrac{6}{x}+\dfrac{10}{y}=4\end{array}\right\}\begin{array}{l}x=2\frac{84}{87}\\[2ex]y=5\frac{1}{17}\end{array}$

20.† $\left.\begin{array}{l}\dfrac{3}{x}-\dfrac{5}{y}=0\\[2ex]\dfrac{6}{x}+\dfrac{10}{y}=4\end{array}\right\}$

## ELIMINATION BY COMPARISON.

**112.** 1. Find the values of $x$ and $y$ in the equations,

$$x + y = 30 \quad (1) \qquad \text{and} \qquad x - y = 6 \quad (2)$$

Transpose $y$ in each of these equations, and we have

$$x = 30 - y \quad (3) \qquad \text{and} \qquad x = 6 + y \quad (4)$$

Now by ax. 1 these values of $x$ must be equal, that is,

By *comparison* $\qquad 30 - y = 6 + y \qquad (5)$

whence $\qquad y = 12 \qquad (6)$

and from (3) or (4) $\qquad x = 18 \qquad (7)$

2. Find the values of $x$ and $y$ in the equations,

$$\frac{4}{x} + \frac{3}{y} = 2 \quad (1) \qquad \text{and} \qquad \frac{8}{x} - \frac{15}{y} = -1 \quad (2)$$

*Solution.*

$$4y + 3x = 2xy \quad (3) \qquad 8y - 15x = -xy \quad (4)$$

$$3x - 2xy = -4y \quad (5) \qquad 15x - xy = 8y \quad (6)$$

$$(3 - 2y)\,x = -4y \quad (7) \qquad (15 - y)\,x = 8y \quad (8)$$

$$x = \frac{-4y}{3 - 2y} \quad (9) \qquad x = \frac{8y}{15 - y} \quad (10)$$

By *comparison* $\quad \dfrac{-4y}{3 - 2y} = \dfrac{8y}{15 - y} \quad (11)$ $\quad$ *vide* **101**, ex. 7.

whence $\qquad y = 4\frac{1}{5}$ and $x = 3\frac{1}{9}$.

It is evident that these steps may be taken on any two equations, hence,

**113.** Having two equations with two unknown quantities, to find their values by the method of *elimination by comparison:*

(1.) If necessary, clear the equations of fractions.

(2.) Find, in both equations, the value of the same unknown quantity, in terms of the other, and make these values equal.

(3.) From the equation thus formed, find the value of the letter involved.

(4.) Same as **110**, 4.

3. $\left.\begin{aligned} 4x + 3y &= 7 \\ x + y &= 2 \end{aligned}\right\}$

4. $\left.\begin{aligned} \frac{x+y}{2} + 25 &= x \\ \frac{x+y}{3} - 5 &= y \end{aligned}\right\}$

5. $\left.\begin{aligned} x + 25 &= 3\tfrac{1}{2}y \\ y + 25 &= \frac{x}{2} + 15 \end{aligned}\right\}$

6. $\left.\begin{aligned} x + \frac{3y}{4} &= 117\tfrac{1}{2} \\ \frac{x}{4} + \frac{3y}{20} &= 27\tfrac{1}{2} \end{aligned}\right\}$

7. $\left.\begin{aligned} \frac{x+y}{2} + \frac{y}{3} &= 8\tfrac{1}{2} \\ x + 2 &= 3\tfrac{1}{4}y \end{aligned}\right\}$

8. $\left.\begin{aligned} x + 2 &= 3\tfrac{1}{4}y \\ y + 4 &= \tfrac{1}{2}x \end{aligned}\right\}$

9. $\left.\begin{aligned} 3x + 2 &= 14y \\ \frac{x+y}{2} &= 2\tfrac{1}{4}x \end{aligned}\right\}$

10. $\left.\begin{aligned} 10x + y &= 4\,(x + y) \\ 10x + y + 18 &= 10y + x \end{aligned}\right\}$

11.† $\left.\begin{aligned} \frac{8}{x} + \frac{5}{y} &= 2 \\ \frac{4}{x} - \frac{5}{3y} &= \tfrac{1}{6} \end{aligned}\right\}$

12.† $\left.\begin{aligned} \frac{1}{3x} + \frac{1}{4y} &= 2 \\ \frac{1}{2x} - \frac{1}{y} &= -2\tfrac{1}{2} \end{aligned}\right\}$

**114.** Either of the three methods of elimination may be employed to solve equations consisting of two unknown quantities. Practice, however, and repeated efforts to do so, will enable the student greatly to abridge the work in almost every case that can happen. To illustrate this remark, we will repeat, by all the methods, the solution of ex. 16 **111.**

1. The equations are $\quad \frac{2}{x} + \frac{2}{y} = 10 \qquad (1)$

and $\quad \frac{1}{x} - \frac{3}{y} = -7 \qquad (2)$

$$\frac{1}{x} + \frac{1}{y} = 5 \qquad (3) \qquad = (1) \div 2$$

$$\frac{4}{y} = 12 \qquad (4) \qquad = (3) - (2)$$

whence $\quad y = \tfrac{1}{3}$ and $x = \tfrac{1}{2}$.

2. After equation (3) we may continue thus:

$$\frac{1}{x} = 5 - \frac{1}{y} \qquad (4) \qquad = (3) \text{ transposed}$$

$$5 - \frac{1}{y} - \frac{3}{y} = -7 \qquad (5) \qquad = (2) \text{ since } \frac{1}{x} = 5 - \frac{1}{y}$$

whence $y = \frac{1}{3}$ and $x = \frac{1}{2}$.

3. Or, we may continue (1), thus:

$$\frac{1}{x} = 5 - \frac{1}{y} \qquad (4)$$

$$\frac{1}{x} = \frac{3}{y} - 7 \qquad (5)$$

By comparison $$5 - \frac{1}{y} = \frac{3}{y} - 7 \qquad (6)$$

whence, again $y = \frac{1}{3}$ and $x = \frac{1}{2}$.

In a similar manner, solve those equations marked † in **111**, **112**, and **113**.

When the coëfficients and signs of the letters $x$ and $y$ interchange, or if the *signs* remain the same in the two equations, we may proceed as follows:

Given $$\frac{4x}{7} + \frac{3y}{8} = 37 \qquad (1)$$

and *vide* **107**, ex. 30 $$\frac{3x}{8} + \frac{4y}{7} = 42\tfrac{1}{2} \qquad (2)$$

$$32x + 21y = 37 \times 56 \qquad (3) \qquad = (1) \times 56$$

$$21x + 32y = 42\tfrac{1}{2} \times 56 \qquad (4) \qquad = (2) \times 56$$

$$53x + 53y = 79\tfrac{1}{2} \times 56 \qquad (5) \qquad = (4) + (3)$$

$$-11x + 11y = 5\tfrac{1}{2} \times 56 \qquad (6) \qquad = (4) - (3)$$

$$x + y = \tfrac{3}{2} \times 56 \qquad (7) \qquad = (5) \div 53$$

$$-x + y = \tfrac{1}{2} \times 56 \qquad (8) \qquad = (6) \div 11$$

whence, $x = 28$ and $y = 56$ *vide* **108**, ex. 1.

In a similar manner, solve the equations marked * in the preceding sections.

## THREE EQUATIONS INVOLVING THREE UNKNOWN QUANTITIES.

**115.** 1. Take the equations,

$$\left.\begin{array}{l} x+\frac{y}{2}+\frac{z}{4}=9 \\ \frac{x}{4}+y+\frac{z}{8}=8 \\ x+2y-3z=-8 \end{array}\right\} \quad \begin{array}{l}(1)\\(2)\\(3)\end{array}$$

$$\begin{array}{rll} \left.\begin{array}{r} 4x+2y+z=36 \\ 2x+8y+z=64 \end{array}\right\} & \begin{array}{l}(4)\\(5)\end{array} & \begin{array}{l}=(1)\times 4\\=(2)\times 8\end{array} \\ -2x+6y=28 & (6) & =(5)-(4) \\ \left.\begin{array}{r} 12x+6y+3z=108 \\ x+2y-3z=-8 \end{array}\right\} & \begin{array}{l}(7)\\(3)\end{array} & \begin{array}{l}=(4)\times 3\\ \end{array} \\ 13x+8y=100 & (8) & =(7)+(3) \\ -13x+39y=182 & (9) & =(6)\times 6\frac{1}{2} \\ 47y=282 & (10) & =(8)+(9) \end{array}$$

Whence, $y=6$, and from (6) $x=4$, and from (4) $z=8$.

Hence, having three equations with three unknown quantities, to find their values:

(1.) If necessary, clear the equations of fractions.

(2.) From any two equations eliminate either of the letters.

(3.) From any other two equations eliminate the same letter.

(4.) Proceed as in **109**, **110**, or **111** with the two equations thus obtained.

EXAMPLES.

2. $\left.\begin{array}{l} x+y+z=18 \\ x+3y+2z=38 \\ x+\frac{y}{3}+\frac{z}{2}=10 \end{array}\right\}$

4. $\left.\begin{array}{r} 2z=21-\frac{1}{3}(x+y) \\ 3x=72 \\ 38=\frac{1}{2}(3x+y-z) \end{array}\right\}$

3. $\left.\begin{array}{l} x+y+2z=9 \\ x+2y+3z=14 \\ 6x+5y+3z=25 \end{array}\right\}$

5. $\left.\begin{array}{l} x+2(y+z)=31 \\ y+3(x+z)=42 \\ z+4(x+y)=51 \end{array}\right\}$

6. $$\left.\begin{aligned} \frac{x}{2}+\frac{y}{3}-\frac{z}{4}&=23\\ \frac{x}{3}-\frac{y}{4}+\frac{z}{2}&=12\\ \frac{x}{4}+\frac{y}{2}-\frac{z}{3}&=17 \end{aligned}\right\}$$

7. $$\left.\begin{aligned} 8x-9y-7z&=-36\\ 12x-y-3z&=36\\ 6x-2y-z&=10 \end{aligned}\right\}$$

## FOUR OR MORE EQUATIONS INVOLVING A LIKE NUMBER OF UNKNOWN QUANTITIES.

**116.** 1. Given, the four equations,

$$\left.\begin{aligned} x+2y+2z+2w&=26\\ 3x+y+3z+3w&=36\\ 4x+4y+z+4w&=44\\ 5x+5y+5z+w&=50 \end{aligned}\right\}\quad\begin{aligned}&(1)\\&(2)\\&(3)\\&(4)\end{aligned}$$

$$\begin{aligned} 3x+6y+6z+6w&=78 &\quad (5)\quad &=(1)\times 3\\ 4x+8y+8z+8w&=104 &\quad (6)\quad &=(1)\times 4\\ 5x+10y+10z+10w&=130 &\quad (7)\quad &=(1)\times 5 \end{aligned}$$

Three equations,

$$\left.\begin{aligned} 5y+3z+3w&=42\\ 4y+7z+4w&=60\\ 5y+5z+9w&=80 \end{aligned}\right\}\quad\begin{aligned}&(8)\quad &=(5)-(2)\\&(9)\quad &=(6)-(3)\\&(10)\quad &=(7)-(4)\end{aligned}$$

$$\begin{aligned} 20y+12z+12w&=168 &\quad (11)\quad &=(8)\times 4\\ 20y+35z+20w&=300 &\quad (12)\quad &=(9)\times 5 \end{aligned}$$

Two equations, $$\left.\begin{aligned} 23z+8w&=132\\ 2z+6w&=38 \end{aligned}\right\}\quad\begin{aligned}&(13)\quad &=(12)-(11)\\&(14)\quad &=(10)-(8)\end{aligned}$$

$$\begin{aligned} 69z+24w&=396 &\quad (15)\quad &=(13)\times 3\\ 8z+24w&=152 &\quad (16)\quad &=(14)\times 4 \end{aligned}$$

One equation, $$61z=244 \quad (17)\quad =(16)-(15)$$

Whence, $z=4$, by (14) $w=5$, by (8) $y=3$, by (1) $x=2$.

It is evident that in the same way we may solve any number of equations involving a number of unknown quantities equal to

that of the equations, *i. e.*, we may always find the value of each letter by the following steps:

(1.) Eliminate any letter by different combinations of two equations till that letter entirely disappears, leaving the number of new equations less by *one*.

(2.) By different combinations of these new equations, eliminate any other letter, till the number of equations is less by *one more*.

(3.) Continue these operations till an equation is obtained containing *one* unknown quantity, and find its value.

(4.) Find the value of the other letters by successive substitutions. *It is not necessary that every letter be found in all the equations.*

EXAMPLES.

2. $$\left.\begin{array}{l} 2x + 3y + 4z + 5w = 40 \\ 3x + 2y - 4z + w = -1 \\ 5x + 4y - 2z + w = 11 \\ 7x - 5y + 3z - w = 2 \end{array}\right\}$$

3. $$\left.\begin{array}{l} 3u + x + 2y - z = 22 \\ 4x - y + 3z = 35 \\ 4u + 3x - 2y = 19 \\ 2u + 4y + 2z = 46 \end{array}\right\}$$

## SYMMETRICAL EQUATIONS.

**117.** The preceding principles will solve any set of equations which can occur. Nevertheless there are many short processes which depend upon the nature of the equations involved. Most of these processes occur in connection with what are called *symmetrical equations*, some examples of which we will now give.

1. Given, to find $x$, $y$, and $z$.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{8} \qquad (1)$$

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{9} \qquad (2)$$

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{10} \qquad (3)$$

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{242}{720} \qquad (4) \qquad =(1)+(2)+(3)$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{121}{720} \qquad (5) \qquad =(4)\div 2$$

$$\frac{1}{x}=\frac{49}{720} \qquad (6) \qquad =(5)-(3)$$

$$\frac{1}{y}=\frac{41}{720} \qquad (7) \qquad =(5)-(2)$$

$$\frac{1}{z}=\frac{31}{720} \qquad (8) \qquad =(5)-(1)$$

Whence, $x=14\frac{34}{49}$, $y=17\frac{23}{41}$, $z=23\frac{7}{31}$.

In a similar manner solve,

2. $$\left.\begin{aligned} x+y&=19\\ x+z&=18\\ y+z&=17\end{aligned}\right\}$$

3. $$\left.\begin{aligned} \frac{2}{x}+\frac{2}{y}&=\frac{7}{6}\\ \frac{2}{x}+\frac{2}{z}&=\frac{16}{15}\\ \frac{2}{y}+\frac{2}{z}&=\frac{9}{10}\end{aligned}\right\}$$

4. $$\left.\begin{aligned} x+y+z&=21\\ x+y+w&=22\\ y+z+w&=24\\ x+z+w&=23\end{aligned}\right\}$$

5. $$\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=\frac{3}{8}\\ \frac{1}{x}+\frac{1}{y}+\frac{1}{w}&=\frac{11}{24}\\ \frac{1}{x}+\frac{1}{z}+\frac{1}{w}&=\frac{1}{2}\\ \frac{1}{y}+\frac{1}{z}+\frac{1}{w}&=\frac{13}{24}\end{aligned}\right\}$$

6. $$\left.\begin{aligned} x+3(y+z)&=30\\ y+3(x+z)&=28\\ z+3(x+y)&=26\end{aligned}\right\}$$

## PROBLEMS
### INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

**118.** All problems must involve as many independent equations as they contain unknown quantities.

EXAMPLES.

1. *The sum of two numbers is 30, and their difference 6 What are the numbers? (*Vide* **108**, ex. 1.)

2. Three times the money of A added to twice that of B would make $22; but twice that of A added to three times that of B would make $23. What is the amount each has in possession? (*Vide* **108**, ex. 2.)

3. *If the first of two numbers be multiplied by 3 and the second by 5, the sum of the products will be 165; but if the first be divided by 4 and the second by 7, the sum of the quotients will be 8. What are the numbers? *Ans.* 20 and 21.

4. If 7 be added to the first of two numbers, the sum will be three times the second; but if 7 be added to the second, the sum will be five times the first. What are the numbers? *Ans.* 2 and 3.

5. *The sum of two numbers is 100, and their difference 20. What are the numbers?

6. *The sum of two numbers is 40, and the greater is three times the less. What are the numbers?

7. Says A to B, "Give me $5 and we shall have equal sums." Now together they have $50. How much does each possess?

8. If a $25 saddle be placed on a horse his value will be twice a second horse; but if the same saddle be placed on the second horse his value will still be $25 less than the first horse. The value of each horse is required.

9. In a mixture of corn and wheat $\frac{1}{2}$ the whole $+ 5$ bushels was corn; but $\frac{1}{3}$ the whole $+ 10$ bushels was wheat. What was the quantity of each?

10. Divide 50 into two parts so that twice the first shall be $\frac{1}{2}$ the second.

11. *Divide 72 into two parts so that $\frac{1}{3}$ the first and $\frac{1}{5}$ the second shall be equal.

12. *Divide 36 into three parts so that $\frac{1}{2}$ the first, $\frac{1}{3}$ the second, and $\frac{1}{4}$ the third may be equal. (*Vide* **104**, ex. 10.)

Let $x$, $y$, and $z$ represent the parts, and $m$ the quantity to which they are to be equal when divided by 2, 3, and 4.

Then $$x + y + z = 36 \qquad (1)$$

and $$\frac{x}{2} = m, \ \frac{y}{3} = m, \ \frac{z}{4} = m$$

Whence $$x = 2m, \ y = 3m, \ z = 4m$$

Adding which $$x + y + z = 9m \qquad (2)$$

By comparing (1) and (2), $9m = 36 \therefore m = 4$, $x = 8$, $y = 12$, and $z = 16$.

13. *The sum of the first and second of three numbers is 11, of the first and third 12, of the second and third 13. What are the numbers? (*Vide* **117**, ex. 2.)

14. *The sum of the reciprocals of the first and second of three numbers is 5; of the first and third 7; of the second and third 8. What are the numbers?

15. A and B have the same income; A saves $\frac{1}{5}$ of his annually; but B by spending \$50 per annum more than A, at the end of six years finds himself \$150 in debt. What is the income of each?

16. *If 8 be added to the numerator of a fraction, the value of the fraction will be 2; but if 8 be added to the denominator, the value will be only $\frac{2}{3}$. Required the fraction.

17. A number expressed by two digits is four times the sum of the digits. If 27 be added to the number, the digits will be interchanged. What is the number?

Let $x =$ the left digit, and $y$ the right.

Then $\qquad 10x + y = 4(x + y) \qquad (1)$

and $\qquad 10x + y + 27 = 10y + x \qquad (2)$

Whence $\qquad 10x + y = 36$ *Ans.* $\qquad$ *Vide* **29**, ex. 1.

18. A number is expressed by three digits. The middle digit is twice the numerical value of the left-hand digit, and is greater by 3 than the right-hand digit. If 99 be subtracted from the number, the right takes the place of the left-hand digit, whilst the middle digit remains the same. What is the number?

19. A number is expressed by four digits. The fourth or left-hand digit is one-half the second. The first or right-hand digit is less than the third by 2. The local value of the fourth is 50 times the local value of the second. If 909 be subtracted from the number, the order of the digits is exactly reversed. What is the number?

20. A number consists of three figures. The left-hand figure is double the right. The sum of the digits is 3. If 81 be subtracted from the number, the left-hand figure is found in the middle, the middle figure is removed to the unit's place, whilst the unit figure appears at the left. What is the number?

21. *A and B can do a piece of work in 8 days; A and C in 9 days; B and C in 10 days. In what time can each do the work alone? And in what time if all work together? (*Vide* **117**, ex. 1.)

22. *A, B, and C can do a piece of work in $2\frac{2}{3}$ hours; B, C, and D in $1\frac{11}{13}$ hours; A, B, and D in $2\frac{2}{11}$ hours; A, C, and D in precisely 2 hours. In what time can each do the work

alone? In what time can all do the work together? (*Vide* **117**, ex. 5.) *Ans.* for all 1*hr.* 56*min.*

23. A bill of $5000 was paid in eagles and half-eagles, using of both kinds 560 pieces. What number of each was used?

24. Says A to B, "Ten years ago I was three times as old as you at that time; now my age is only double yours." What is the age of each?

25. If 5 times A's property is added to $\frac{1}{5}$ of B's, the sum will be $2700. If 5 times B's is added to $\frac{1}{5}$ of A's, the sum will be $5100. What is the property of each?

26. I have a gold and a silver watch, and a chain worth $50. If the chain be attached to the silver watch, together they are worth $\frac{1}{2}$ the gold watch. But when the chain is worn with the gold watch, they are worth 5 times the silver watch. The value of each watch is required.

27. At an election the majority was 80 votes. Had $\frac{1}{4}$ the minority votes and 25 votes more been given to the successful candidate, he would have received in all double the number of his opponent. How many votes were actually given to each candidate?

28. The crown of Hiero, king of Syracuse, weighed 20 pounds in air and $18\frac{3}{4}$ pounds in water. Now $19\frac{16}{25}$ pounds of gold weigh $18\frac{16}{25}$ pounds in water, and $10\frac{1}{2}$ pounds of silver weigh $9\frac{1}{2}$ pounds in water. How much gold and how much silver did the crown contain? *Ans.* gold 14.77, silver 5.23.

29. *If a grocer mix sherry and brandy in the ratio of 2 to 1, the mixture is worth 78 shillings per dozen. If he mix them in the ratio of 7 to 8, the mixture is worth 79 shillings per dozen. What is the price per dozen of each kind of wine?

30. In a composition of gunpowder the nitre was 10 pounds more than $\frac{2}{3}$ of the whole, the sulphur $4\frac{1}{2}$ pounds less than $\frac{1}{6}$ of the whole, and the charcoal 2 pounds less than $\frac{1}{2}$ the nitre. What was the quantity of powder?

31. A vintner sold at one time 20 dozen of port wine and 30 dozen of sherry for \$120. At another time, 30 dozen of port and 25 dozen of sherry, by a rise of \$1 each per dozen, brought \$195. What was the first price of each per dozen?

32. A's property together with $\frac{1}{2}$ B's and C's is worth \$3500; B's with $\frac{1}{3}$ A's and C's is worth \$5000; C's with $\frac{1}{4}$ A's and B's is worth \$5250. What is the property of each?

33. On examining my watch I find that $\frac{1}{2}$ the time past noon is $\frac{5}{26}$ of the time till midnight. What is the time?

34. A farmer has 30 bushels of oats, at 30¢ per bushel, which he wishes to mix with corn at 70¢ and barley at 90¢ per bushel, making a mixture of 200 bushels, at 80¢ per bushel. How much corn and barley must he mix with the oats?

35. A farmer has 86 bushels of wheat, at 4*s.* 6*d.* per bushel. Barley, at 3*s.* per bushel, and rye, at 3*s.* 6*d.* per bushel, are to be mixed with the wheat so as to make a mixture of 136 bushels worth 4*s.* per bushel. How much rye and barley are to be used?

36. A gentleman left a sum of money to be divided among four servants. The share of the first was $\frac{1}{2}$ the sum of the shares of the other three. The share of the second was $\frac{1}{3}$ the sum of the other three. The share of the third was $\frac{1}{4}$ the sum of the other three. The share of the first exceeded that of the last by \$14. What was the amount divided and the share of each?

37. A person pays at one time two creditors \$53, giving to one $\frac{4}{11}$ the sum due him, and to the other \$3 over $\frac{1}{6}$ the sum due him. At another time he pays the two \$42, giving the first $\frac{3}{7}$ of what remains due him, and the second $\frac{1}{3}$ of what remains due him. How much did he owe each?

38. Three persons, A, B, and C, have \$96 among them. A gives to B and C as much as they already have. Then B gives to A and C as much as they have; after which C gives to A and B as much as they then have. After this distribution each has \$32. How much did each have at first?

39. *A person has two kinds of money; it takes 10 pieces of one to make a dollar, and 2 pieces of the other to make the same sum. Some one offers him a dollar for 6 pieces, if he could make the change even. How many pieces were used of each kind?

40. *A man had dimes and half-dollars, and paid a debt of \$2 with 12 pieces. How many of each kind did he use?

41. A man has eagles and half-eagles, and pays a debt of \$65 with 8 pieces. How many of each kind must be used?

42. A man has 100 dimes and half-dimes, out of which he paid a debt of \$6.15, and had 27 pieces left. How many dimes and half-dimes were taken?

## LITERAL EQUATIONS.

**119.** 1. Given, the equations,

$$\frac{a}{x} + \frac{b}{y} = m \qquad (1)$$

and

$$\frac{b}{x} + \frac{c}{y} = n \qquad (2)$$

$$\frac{ab}{x} + \frac{b^2}{y} = bm \qquad (3) \qquad = (1) \times b$$

$$\frac{ab}{x} + \frac{ac}{y} = an \qquad (4) \qquad = (2) \times a$$

$$\frac{b^2}{y} - \frac{ac}{y} = bm - an \qquad (5) \qquad = (3) - (4)$$

Whence

$$y = \frac{b^2 - ac}{bm - an} \qquad (6)$$

$$\frac{ac}{x} + \frac{bc}{y} = cm \qquad (7) \qquad = (1) \times c$$

$$\frac{b^2}{x} + \frac{bc}{y} = bn \qquad (8) \qquad = (2) \times b$$

$$\frac{b^2}{x} - \frac{ac}{x} = bn - cm \qquad (9) \qquad = (8) - (7)$$

Whence

$$x = \frac{b^2 - ac}{bn - mc} \qquad (10)$$

2. Given, $ax + by = m$ and $cx + dy = n$, to find $x$ and $y$.

*Ans.* $x = \dfrac{dm - bn}{ad - bc}$, $y = \dfrac{an - cm}{ad - bc}$.

3. Given, $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $x + y = c$, to find $x$ and $y$.

*Ans.* $x = \dfrac{(c - b)\,a}{a - b}$, $y = \dfrac{(a - c)\,b}{a - b}$.

4. Given, $\dfrac{1}{x} + \dfrac{1}{y} = a$, $\dfrac{1}{x} + \dfrac{1}{z} = b$, & $\dfrac{1}{y} + \dfrac{1}{z} = c$, to find $x$, $y$, $z$.

*Ans.* $x = \dfrac{2}{a + b - c}$, $y = \dfrac{2}{a - b + c}$, $z = \dfrac{2}{-a + b + c}$.

5. Given, $\frac{1}{x}+\frac{1}{y}=\frac{2}{a}$, $\frac{1}{x}+\frac{1}{z}=\frac{2}{b}$, & $\frac{1}{y}+\frac{1}{z}=\frac{2}{c}$, to find $x$, $y$, $z$.

*Ans.* $x=\frac{abc}{bc+ac-ab}$, $y=\frac{abc}{bc-ac+ab}$, $z=\frac{abc}{-bc+ac+ab}$.

6. Given, $\frac{a}{x}+\frac{b}{y}=m$ and $\frac{c}{x}+\frac{d}{y}=n$, to find $x$ and $y$.

*Ans.* $x=\frac{ad-bc}{dm-bn}$, $y=\frac{ad-bc}{an-cm}$.

7. Given, $\frac{x}{a}+\frac{y}{b}=m$ and $\frac{x}{c}+\frac{y}{d}=n$, to find $x$ and $y$.

*Ans.* $x=\frac{(dn-bm)\,ac}{ad-bc}$, $y=\frac{(am-cn)\,bd}{ad-bc}$.

8. Given, $ax+by=m$ and $bx+ay=m$, to find $x$ and $y$.

*Ans.* $x=\frac{m}{a+b}$, $y=\frac{m}{a+b}$.

9. Given, $\frac{1}{x}+\frac{1}{y}=\frac{1}{c}$ and $ax=by$, to find $x$ and $y$.

*Ans.* $x=\frac{(a+b)\,c}{a}$, $y=\frac{(a+b)\,c}{b}$.

## GENERALIZATIONS.

**120.** 1. A and B together can do a piece of work in $c$ days. The time in which A can do the work is to the time in which B can do it alone as $b$ is to $a$. In what time can each do it alone? The equations of this problem are those of ex. 9, **119.**

Hence, A requires $\frac{(a+b)\,c}{a}$, and B $\frac{(a+b)\,c}{b}$.

If $c=20$, $a=1$, $b=2$, then A requires 60, and B 30 days.

2. A person has two kinds of money: it takes $a$ pieces of the first to make a dollar, and $b$ pieces of the second to make the same sum. Some one offers him a dollar for $c$ pieces. How many of each kind did it take? The equations of this problem are those of ex. 3, **119.**

Hence it takes $\frac{(c-b)\,a}{a-b}$ and $\frac{(a-c)\,b}{a-b}$.

If $a = 10$, $b = 2$, $c = 6$, the problem is 39*, **118.**

If $a = 6\frac{1}{2}$, $b = 13$, $c = 8$, the problem is 41, **118.**

Reading \$65 for *a dollar.*

If $a = 61\frac{1}{2}$, $b = 123$, $c = 73$, the problem is 42, **118.**

Reading \$6.15 for *a dollar.*

3. If a grocer mix sherry and brandy, in the ratio of $a$ to $b$, the mixture is worth $m$ dollars per dozen. If he mix in the ratio of $c$ to $d$, the mixture is worth $n$ dollars per dozen. What is the price of each per dozen?

*Ans.* Sherry $\dfrac{(a+b)\,dm-(c+d)\,bn}{ad-bc}$, Brandy $\dfrac{(c+d)\,an-(a+b)\,cm}{ad-bc}$

If $a = 2$, $b = 1$, $m = 78$, $c = 7$, $d = 2$, $n = 79$, the problem is 29, **118.** *Vide* **28**, ex. 13.

4. The sum of two numbers is $a$, and their difference $b$. What are the numbers? *Ans.* $x = \dfrac{a+b}{2}$, $y = \dfrac{a-b}{2}$.

If $a = 30$ and $b = 6$, the problem is ex. 1, **118.**

If $a = 100$ and $b = 20$, the problem is ex. 5, **118.**

In the same manner generalize the problems marked * in **118.**

5. Find $x$ and $y$ in the equation,

$$x + y = c \tag{1}$$

and

$$y^2 + b^2 = x^2 + a^2 \tag{2}$$

since $x = c - y$, we have

$$y^2 + b^2 = (c - y)^2 + a^2 \tag{3}$$

that is,

$$y^2 + b^2 = c^2 + 2cy + y^2 + a^2 \tag{4}$$

Whence, $y = \dfrac{c^2 + a^2 - b^2}{2c}$ and $x = \dfrac{c^2 + b^2 - a^2}{2c}$.

6. A's property, together with $l$ times what B and C are worth, is equal to $p$ dollars. B's property, together with $m$ times what A and C are worth, is equal to $q$ dollars. C's property, together with $n$ times what A and B are worth, is equal to $r$ dollars. What is the property of each?

*Solution:—*

$$\left.\begin{aligned} x + l\,(y+z) &= p \\ y + m\,(x+z) &= q \\ z + n\,(x+y) &= r \end{aligned}\right\}\qquad \begin{aligned}&(1)\\&(2)\\&(3)\end{aligned}$$

$$x - lx + lx + ly + lz = p \qquad (4)$$

$$x\,(1-l) + l\,(x+y+z) = p \qquad (5)$$

$$\left.\begin{aligned} x + \frac{l}{1-l}(x+y+z) &= \frac{p}{1-l} \\ y + \frac{m}{1-m}(x+y+z) &= \frac{q}{1-m} \\ z + \frac{n}{1-n}(x+y+z) &= \frac{r}{1-n} \end{aligned}\right\}\qquad \begin{aligned}&(6)\\&(7)\\&(8)\end{aligned}$$

$$x+y+z+\left(\frac{l}{1-l}+\frac{m}{1+m}+\frac{n}{1-n}\right)(x+y+z) = \frac{p}{1-l}+\frac{q}{1-m}+\frac{r}{1-n}$$

$$(9) \qquad = (6) + (7) + (8)$$

$$\left(1+\frac{l}{1-l}+\frac{m}{1-m}+\frac{n}{1-n}\right)(x+y+z) = \frac{p}{1-l}+\frac{q}{1-m}+\frac{r}{1-n}$$

$$(10) \qquad = (9) \text{ factored}$$

$$x+y+z = \frac{\dfrac{p}{1-l}+\dfrac{q}{1-m}+\dfrac{r}{1-n}}{1+\dfrac{l}{1-l}+\dfrac{m}{1-m}+\dfrac{n}{1-n}} \qquad (11)$$

$$x = \frac{p}{1-l} - \frac{l}{1-l}\left\{\frac{\dfrac{p}{1-l}+\dfrac{q}{1-m}+\dfrac{r}{1-n}}{1+\dfrac{l}{1-l}+\dfrac{m}{1-m}+\dfrac{n}{1-n}}\right\} \qquad (12)$$

$$y = \frac{q}{1-m} - \frac{m}{1-m}\left\{\frac{\dfrac{p}{1-l}+\dfrac{q}{1-m}+\dfrac{r}{1-n}}{1+\dfrac{l}{1-l}+\dfrac{m}{1-m}+\dfrac{n}{1-n}}\right\} \qquad (13)$$

$$z = \frac{r}{1-n} - \frac{n}{1-n}\left\{\frac{\dfrac{p}{1-l}+\dfrac{q}{1-m}+\dfrac{r}{1-n}}{1+\dfrac{l}{1-l}+\dfrac{m}{1-m}+\dfrac{n}{1-n}}\right\} \qquad (14)$$

Equation (4) is obtained by subtracting and adding $lx$ from the first member of (1). Equation (12) is obtained from (6) by transposition and the substitution of the value of $x + y + z$. Equation (7) is obtained from (2) by taking the same steps as were taken on (1), or, what is much better, (7) may be obtained from (6) by writing an equation of the exact form of (6), but using the next letters in the alphabet, returning to the *first* letters when all in the original equations are exhausted. Equation (8) is obtained from (7), and (13) from (12), also (14) from (13) in the same way. (*Vide* **117**, 1.)

If $l = 2$, $m = 3$, $n = 4$, $p = 56$, $q = 77$, $r = 96$, then $x = 10$, $y = 11$, $z = 12$.

If $l = \frac{1}{2}$, $m = \frac{1}{3}$, $n = \frac{1}{4}$, $p = 3500$, $q = 5000$, $r = 5250$, then $x = 2000$, $y = 3000$, $z = 4000$. (*Vide* **118**, ex. 32.)

## NEGATIVE RESULTS.

**121.** 1. A wishes to pay a debt with 100 dimes and half-dimes, the debt being \$4. How many of each are required?

Let $x$ = the dimes, and $y$ = the half-dimes.

Then from the question $\quad x + y = 100 \quad (1)$

and $\quad 10x + 5y = 400 \quad (2)$

Whence, $\quad x = -20$, and $y = 120$.

From this answer we discover that it is impossible to pay \$4 by the use of exactly 100 dimes and half-dimes.

A must pay 120 half-dimes, and receive back 20 dimes in change. Had the question read,

1.* In paying a debt of \$4, A gave 100 more half-dimes than he received dimes in change. What number of each passed between the parties?

We should then have, $\quad y - x = 100 \quad (1)$

and $\quad 5y - 10x = 400 \quad (2)$

Whence, $\quad x = 20$, and $y = 120$.

The results are now positive, showing the problem to be arithmetically possible. (*Vide* **51** *et ante.*)

In general, if in the solution of a problem a negative result is obtained, we conclude that the problem as enunciated involves an arithmetical impossibility. But the results, whatever they may be, are interpreters of the error, and guide to a proper enunciation of the problem.

2. The sum of two numbers is 20, and five times the one added to six times the other makes the sum of 25. What are the numbers? *Ans.* $x = 95$, $y = -75$.

The result shows that the problem should have read,

2.* The *difference* of two numbers is 20, and five times the greater *diminished* by six times the less makes a remainder of 25; for if we substitute $x$ and $-y$ into the equations $x + y = 20$ and $5x + 6y = 25$, they become $x - y = 20$ and $5x - 6y = 25$; in the last two of which $x = 95$ and $y = 75$.

3. If 1 be added to the numerator of a fraction, its value will be $\frac{2}{7}$; but if 1 be added to the denominator, the value will be $\frac{1}{2}$. What is the fraction?

$$\text{Let } \frac{x}{y} = \text{the fraction.}$$

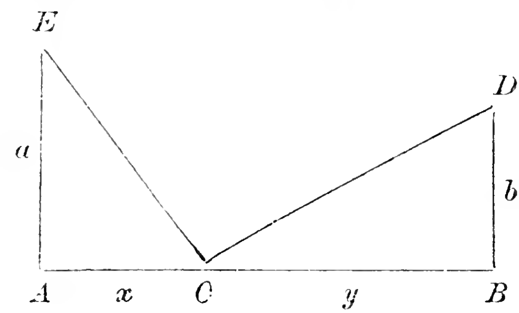
$$\text{Then} \qquad \frac{x+1}{y} = \frac{2}{7} \qquad \text{and} \qquad \frac{x}{y+1} = \frac{1}{2}$$

$$\text{Whence} \qquad x = -3 \qquad \text{and} \qquad y = -7$$

The fraction is therefore $\frac{-3}{-7}$, which must not be interpreted as $\frac{+3}{+7}$. The problem should have read,

3.* If 1 be *subtracted* from the numerator of a fraction, its value will be $\frac{2}{7}$; but if 1 be subtracted from the denominator, the value will be $\frac{1}{2}$. Here $x = 3$, $y = 7$, and the fraction is $\frac{3}{7}$.

4. Two trees, $a$ and $b$ feet high, are situated $c$ feet from each other on a horizontal plain. At what point *between* the trees must a man stand to be equally distant from the top of each?




Suppose the point to be $C$:

Let $x =$ the distance from $A$ to $C$,

and $y =$ the distance from $B$ to $C$.

Now, by the Pythagorean Proposition, Euclid, book I, 47,

$$DC^2 = BC^2 + BD^2 \text{ and } EC^2 = AC^2 + AE^2.$$

But by the question, $DC = EC$, or axiom 7, $DC^2 = EC^2$.

Therefore, axiom 1, $BC^2 + BD^2 = AC^2 + AE^2$.

That is, $y^2 + b^2 = x^2 + a^2$ (1)

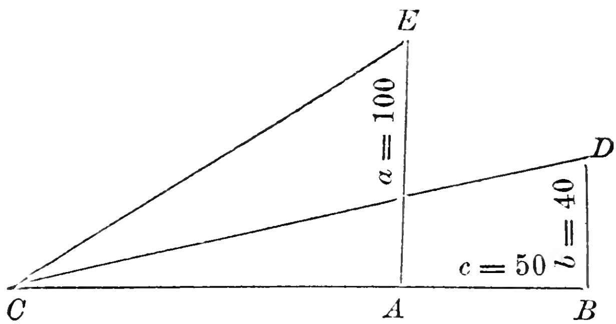
But, $x + y = c$ (2)

Whence, $x = \dfrac{c^2 + b^2 - a^2}{2c}$ and $y = \dfrac{c^2 + a^2 - b^2}{2c}$. *vide* **120**, ex. 5.

I. If $a = 80$, $b = 60$, and $c = 100$, then $x = 36$, $y = 64$. *Vide* **104**, ex. 14.

II. If $a = 100$, $b = 40$, and $c = 50$, then $x = -59$, $y = 109$.

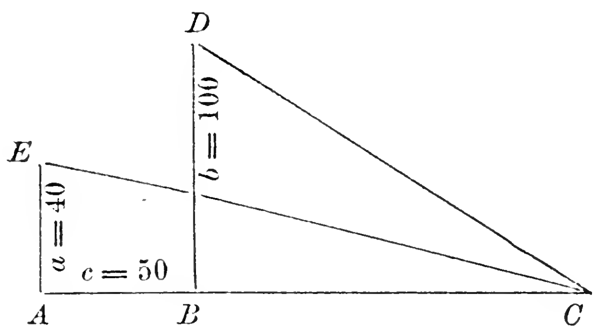
This shows that with these values of $a$, $b$, and $c$ it would be impossible to stand *between* the trees and be equally distant from the top of each. In this case the value of $x$ must be taken *to the left* of $A$ on the prolongation of $BA$, as in the following figure.




In general, then, if lines to the *right* of a point are considered *positive*, those to the *left* will be *negative*. How should the problem have read?

III. If $a = 40$, $b = 100$, and $c = 50$, then $x = 109$ and $y = -59$.

This shows an impossibility similar to the last. This case causes $y$ to be taken to the right of the point $B$, on the prolongation of $A\ B$, as shown by the figure:




If, then, lines to the *left* of a point are considered *positive*, those to the *right* will be *negative*. How should the problem have read?

*Any* supposition which makes $x$ or $y$ negative must be interpreted in like manner.

**122.** INTERPRETATION OF THE SYMBOLS,

$$\frac{0}{A} \qquad \frac{A}{0} \qquad \frac{0}{0}$$

where $A$ represents any finite quantity.

The formulae of the last problem are,

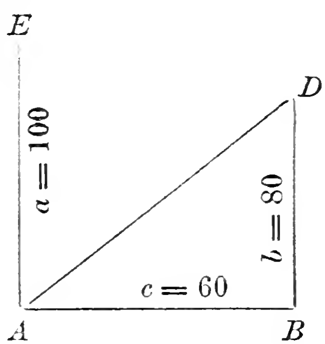
$$x = \frac{c^2 + b^2 - a^2}{2c} \quad \text{and} \quad y = \frac{c^2 + a^2 - b^2}{2c}$$

1. If $a = 100$, $b = 80$, $c = 60$, then the formulae reduce to,

$$x = \frac{3600 + 6400 - 1000}{120} = \frac{0}{120} \quad \text{and}$$

$$y = \frac{10000 + 3600 - 6400}{120} = 60$$

This value of $y$ shows that the point is at the foot of the taller tree, and therefore the expression $\frac{0}{120}$ must be infinitely small. This is illustrated by the figure,




*Any* supposition which makes $c^2 + b^2 = a^2$ must make $y = c$, for if, in the value of $y$ above, we should insert $c^2 + b^2$ for $a^2$, we have,

$$y = \frac{c^2 + c^2 + b^2 - b^2}{2c} = c$$

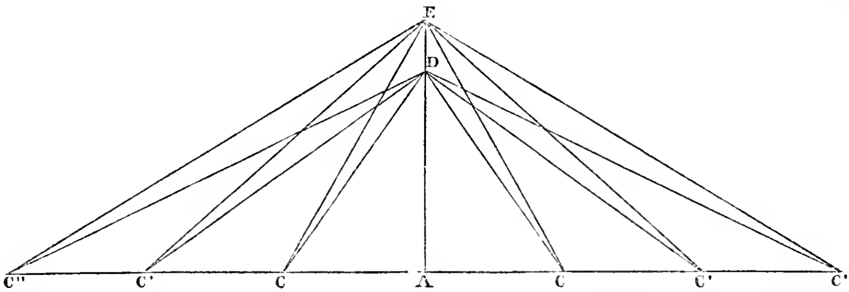
Hence, we assume that $\frac{0}{A}$ is the representative of an infinitely small quantity; that is, the following equation is true.

$$\frac{0}{A} = 0 \qquad (1)$$

II. If $a = 100$, $b = 80$, $c = 0$, then the formulae reduce to,

$$x = \frac{0 + 6400 - 10000}{0} = \frac{-3600}{0} \text{ and}$$

$$y = \frac{0 + 10000 - 6400}{0} = \frac{3600}{0}$$




By the supposition the tree $A$ occupies the same spot with the tree $B$, differing only in height. It is clear that as the point $C$ recedes from $A$, the distance to the top of each tree approaches nearer an equality, and the two distances are absolutely equal only when $C$ is INFINITELY removed from $A$. Hence the expression $\frac{-3600}{0}$ or $\frac{3600}{0}$ must be *infinitely great*, as each of them is the representative of the same distance on different sides of the point $A$. The supposition of $c = 0$ and $a >$ or $< b$ may always be reasoned upon in the same way. Hence $\frac{A}{0}$ represents an infinitely large quantity, that is,

$$\frac{A}{0} = \infty \qquad (2)$$

III. If $a = 80$, $b = 80$, $c = 0$, the formulae reduce to,

$$x = \frac{0 + 6400 - 6400}{0} = \frac{0}{0} \quad \text{and} \quad y = \frac{0 + 6400 - 6400}{0} = \frac{0}{0}$$

By the supposition the trees must be absolutely identical, and it is clear that the point $C$ may be *any where* on the line passing through its base. Hence the symbol is one of *indetermination*, That is,

$$\frac{0}{0} = 1,\ 2,\ 3,\ -1,\ -2,\ -3,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \&c. \qquad (3)$$

**123.** 1. A boy bought apples and oranges, giving 3 cents each for apples and 4 cents each for oranges. How many can he buy for 100 cents?

*Solution.*

From the question we can have only one equation, viz:

$$3x + 4y = 100 \qquad (1)$$

Hence, $$x = \frac{100 - 4y}{3} \qquad (2)$$

Place $y$ for $x$ in (1), and we have,

$$100 - 4y + 4y = 100 \qquad (3)$$

Then, $$(4 - 4)\,y = 100 - 100 \qquad (4)$$

$$y = \frac{0}{0} \qquad (5)$$

Hence, the boy may purchace any number of either he pleases. If, however, none of the fruit is to be cut, the limitation gives rise to one of the most interesting departments of algebra, known as INDETERMINATE ANALYSIS.

EXAMPLES.

2. Find $x$ and $y$ in whole numbers in the equation $3x + 4y = 100$.

We have, $$x = \frac{100 - 4y}{3} = 33 - y + \frac{1 - y}{3}.$$

Since $33 - y$ is a whole number, $\frac{1 - y}{3}$ must be a whole number also.

Let $$\frac{1 - y}{3} = p$$

then $$y = 1 - 3p \qquad 1)$$

and $$x = 33 - (1 - 3p) + p \qquad = 33 + 4p - 1$$

or $$x = 32 + 4p \qquad (2)$$

Then $p$ may be any whole number whatever that will render $x$ and $y$ positive in equations (1) and (2). It is evident from

(1) that $p$ must be 0, or negative. It is evident from (2) that $p$ must be the following quantities, viz:

$$\begin{array}{lrrrrrrrrr} & 0, & -1, & -2, & -3, & -4, & -5, & -6, & -7, & -8. \\ \text{Whence,} \quad x = & 32, & 28, & 24, & 20, & 16, & 12, & 8, & 4, & 0. \\ y = & 1, & 4, & 7, & 10, & 13, & 16, & 19, & 22, & 25. \end{array}$$

Hence, the boy in example 1 could have bought any of these combinations of apples and oranges.

3. Find $x$ and $y$ in whole numbers in the equation,

$$11x + 5y = 254$$

*Ans.* $x = 19, \quad 14, \quad 9, \quad 4.$
$y = 9, \quad 20, \quad 31, \quad 42.$

As this subject is not strictly elementary in its nature, we shall not pursue it further.

# CHAPTER VI.

## INVOLUTION.

**124.**

(1.) The *first* power of a quantity is the quantity itself.

(2.) The *second* power of a quantity is the quantity multiplied by itself.

(3.) The *third* power of a quantity is the quantity multiplied by the 2*nd* power.

(4.) The $m^{th}$ power of a quantity is the quantity multiplied by the $(m-1)^{th}$ power.

(5.) Involution investigates the method of finding any power of a quantity.

(6.) If we multiply $3a$ by $3a$ we have $9a^2$; and $4a^2$ multiplied by $4a^2$ gives $16a^4$; and $7x^3 \times 7x^3 \times 7x^3 = 349x^9 =$ the cube of $7x^3$. Hence,

**125.** To raise a positive monomial to any required power:

*Raise the coëfficient to the required power by multiplication, and multiply the exponents of the letters by the number expressing the power to be obtained.*

EXAMPLES.

1. Find the 2*nd* power of $3a^4$. *Ans.* $9a^8$.
2. Find the 3*rd* power of $4a^5$. *Ans.* $64a^{15}$.
3. Find the 6*th* power of $2x^2y$. *Ans.* $64x^{12}y^6$.
4. Find the 8*th* power of $3x^2y^4$. *Ans.* $6561x^{16}y^{32}$.

5. Find the squares of $2x$, $3x^2y$, $4xy^2$, $5x^2y^2$, $6x^3y$, $7xy^3$, and $8x^4y^4$.

6. Find the cubes of $2x^2$, $5xy^3$, $7x^3y$, $8x^2y^3$, $9xy^5$, $10xy^6$, and $12x^2y^2z^2$.

7. If we multiply $-x$ by $-x$ the product is $+x^2$, and if we again multiply $+x^2$ by $-x$, the product is $-x^3$ and $-x^3 \times -x = +x^4$, &c. But $-x$ = 1*st* power of $-x$, and $+x^2$ = 2*nd* power of $-x$, and $-x^3$ = 3*rd* power of $-x$, and $+x^4$ = 4*th* power of $-x$. Hence,

**126.** The *odd* powers of a negative monomial are *negative*, the *even* powers *positive*.

EXAMPLES.

1. Find the 3*rd* power of $-4a^2x$. *Ans.* $-64a^6x^3$.
2. Find the 2*nd* power of $-16xy$. *Ans.* $256x^2y^2$.
3. Find the 2*nd* power of $-5x^2y^2$. *Ans.* $25x^4y^4$.
4. Find the cube of $-5x^2y$. *Ans.* $-125x^6y^3$.
5. Find the 14*th* power of $-axy$. *Ans.* $a^{14}x^{14}y^{14}$.
6. Find the 9*th* power of $-x^4y^4$. *Ans.* $-x^{36}y^{36}$.
7. Find the 2*nd* power of $2x^{\frac{1}{2}}y^{\frac{1}{2}}$. *Ans.* $4xy$.
8. Find the 3*rd* power of $3x^{\frac{1}{3}}y^{\frac{2}{3}}$. *Ans.* $27xy^2$.
9. Find the 4*th* power of $5x^{\frac{1}{4}}y^{\frac{3}{4}}$. *Ans.* $625xy^3$.
10. Find the 5*th* power of $-4x^{\frac{1}{10}}y^{\frac{1}{15}}$. *Ans.* $-1024x^{\frac{1}{2}}y^{\frac{1}{3}}$.
11. Find the 10*th* power of $-2x^{10}y^{\frac{1}{10}}$. *Ans.* $1024x^{100}y$.
12. Find the 7*th* power of $-2x^{\frac{1}{7}}y^{3\frac{1}{2}}$. *Ans.* $-128xy^{\frac{49}{2}}$.
13. Find the 6*th* power of $-2x^{\frac{1}{6}}y^{\frac{1}{6}}$. *Ans.* $64xy$.
14. Find the 4*th* power of $25xy$. *Ans.* $390625x^4y^4$.
15. Find the squares of $-x^5$, $x^5y$, $3xy^2$, $-4x^3y$, $5xy^3$, $x^{\frac{1}{2}}y^{\frac{1}{2}}$, $x^{\frac{1}{4}}y^{\frac{1}{4}}$, and $x^3$.
16. Find the cubes of $-x^5$, $x^3y^{\frac{1}{3}}$, $5x^{\frac{1}{9}}y^9$, $-x^{\frac{1}{3}}y^2$, $-7xy^{\frac{2}{9}}$, $8xy$, and $-x^7$.
17. Find the 2*nd* power of $a^x$. *Ans.* $a^{2x}$.

18. Find the $m^{th}$ power of $a^x$. *Ans.* $a^{mx}$.

19. Find the $\frac{1}{2}$ power of $a^x$. (*Vide* § **13.**) *Ans.* $a^{\frac{x}{2}}$.

20. Find the $\frac{1}{m}th$ power of $a^x$. (*Vide* § **13.**) *Ans.* $a^{\frac{x}{m}}$.

If $a = 16$, $x = 3$, $m = 4$, then $a^{\frac{x}{m}} = 16^{\frac{3}{4}} = 8$.

**127.** To raise a monomial fraction to any required power:

*Observe if the fraction is in its lowest terms; if not, reduce it, and then raise the numerator and denominator separately to the required power.*

1. Find the 5*th* power of $\dfrac{32x^4y^3}{64x^7y^2} = \dfrac{y}{2x^3}$ *Ans.* $\dfrac{y^5}{32x^{15}}$.

2. Find the 8*th* power of $\dfrac{x^2y^4}{x^3y}$. *Ans.* $\dfrac{y^{24}}{x^8}$.

3. Find the 5*th* power of $\dfrac{x^2y}{-2xy}$. *Ans.* $\dfrac{-x^5}{32}$.

4. Find the 4*th* powers of $\dfrac{-2x^7}{-3x^6}$, $-\dfrac{x^{\frac{1}{4}}}{2y}$, $\dfrac{x^{\frac{7}{2}}}{2x^{\frac{3}{4}}}$, and $\dfrac{8x^{\frac{1}{4}}y}{4xy^{\frac{1}{4}}}$.

5. Find the 6*th* powers of $\dfrac{x}{-y^{\frac{1}{6}}}$, $\dfrac{17x^2y}{34xy}$, $\dfrac{-x^{\frac{1}{3}}}{-y^{\frac{1}{3}}}$, and $\dfrac{44x^{\frac{1}{2}}}{55y^{\frac{1}{2}}}$.

6. Find the 2*nd* power of $\dfrac{a^x}{y^x}$. *Ans.* $\dfrac{a^{2x}}{y^{2x}}$.

7. Find the $m^{th}$ power of $\dfrac{a^x}{y^x}$. *Ans.* $\dfrac{a^{mx}}{y^{mx}}$.

**128.** To raise the positive binomial $x + y$ to any required power:

*Multiply the binomial as indicated in* § **124.**

1. Find the 2*nd*, 3*rd*, 4*th*, &c. powers of $x + y$.

*Solution.*

$$
\begin{array}{lllllll}
\text{Multiply} & x + y & & & & & = \left\{ \begin{array}{l} 1st\ \text{power.} \\ (\mathbf{124},\ 1.) \end{array} \right. \\
\text{by} & x + y & & & & & \\
\hline
 & x^2 + & xy & & & & \\
 & & xy & + \ y^2 & & & \\
\hline
\text{and we have} & x^2 + & 2xy & + \ y^2 & & & = \left\{ \begin{array}{l} 2nd\ \text{power.} \\ (\mathbf{124},\ 2.) \end{array} \right. \\
\text{Multiply the } 2nd \text{ power by} & x + y & & & & & \\
\hline
 & x^3 + & 2x^2y + & xy^2 & & & \\
 & & x^2y + & 2xy^2 & + \ y^3 & & \\
\hline
\text{and we have} & x^3 + & 3x^2y + & 3xy^2 & + \ y^3 & & = \left\{ \begin{array}{l} 3rd\ \text{power.} \\ (\mathbf{124},\ 3) \end{array} \right. \\
\text{Multiply the } 3rd \text{ power by} & x + y & & & & & \\
\hline
 & x^4 + & 3x^3y + & 3x^2y^2 + & xy^3 & & \\
 & & x^3y + & 3x^2y^2 + & 3xy^3 & + \ y^4 & \\
\hline
\text{and we have} & x^4 + & 4x^3y + & 6x^2y^2 + & 4xy^3 & + \ y^4 & = \left\{ \begin{array}{l} 4th\ \text{power.} \\ (\mathbf{124},\ 4.) \end{array} \right. \\
\text{Multiply the } 4th \text{ power by} & x + y & & & & & \\
\hline
 & x^5 + & 4x^4y + & 6x^3y^2 + & 4x^2y^3 + & xy^4 & \\
 & & x^4y + & 4x^3y^2 + & 6x^2y^3 + & 4xy^4 + & y^5 \\
\hline
\end{array}
$$

and we have $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 = 5th$ power.

1. By the mere observation of either of the above powers, the *law* which governs the exponents of both letters is readily discovered.

*The exponents of the first and last terms are indicated by the* POWER ITSELF.

The exponents of $x$ *decrease* towards the right by unity.

The exponents of $y$ *increase* towards the right by unity.

Therefore $x$ disappears from the last term, and $y$ cannot be found in the first. (*Vide* **72**, 3.)

Disregarding the coëfficients, the several powers of $x + y$ may easily be written thus:

$$
\begin{array}{ll}
x + y & = 1st \text{ power.} \\
x^2 + xy + y^2 & = 2nd \text{ power.} \\
x^3 + x^2y + xy^2 + y^3 & = 3rd \text{ power.} \\
x^4 + x^3y + x^2y^2 + xy^3 + y^4 & = 4th \text{ power.} \\
x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5 & = 5th \text{ power.} \\
x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6 & = 6th \text{ power.}
\end{array}
$$

II. When any coëfficient is given, the coëfficient of the *next* term can be found by the following

RULE.

*Multiply the given coëfficient by the exponent of the leading letter in the same term, and divide the product by the number expressing the place of the term counting from the left.*

2. Let us insert the coëfficients of the fifth power.

The coëfficient of the first term must always be *one*, and we write down $\overset{(1)}{x^5}$ as this first term.

By the rule, the coëfficient of the $2nd$ term is $\frac{1 \times 5}{1} = 5$, and we write $\overset{(1)}{x^5} + \overset{(2)}{5x^4y}$.

Again, by the rule, the coëfficient of the $3rd$ term is $\frac{5 \times 4}{2} = 10$, and we write $\overset{(1)}{x^5} + \overset{(2)}{5x^4y} + \overset{(3)}{10x^3y^2}$.

Again, the coëfficient of the $4th$ term is $\frac{10 \times 3}{3} = 10$, and we write $\overset{(1)}{x^5} + \overset{(2)}{5x^4y} + \overset{(3)}{10x^3y^2} + \overset{(4)}{10x^2y^3}$.

The coëfficient of the $5th$ term is $\frac{10 \times 2}{4} = 5$, and we write

$$\overset{(1)}{x^5} + \overset{(2)}{5x^4y} + \overset{(3)}{10x^3y^2} + \overset{(4)}{10x^2y^3} + \overset{(5)}{5xy^4}.$$

Finally, the coëfficient of the $6th$ term is $\frac{5 \times 1}{5} = 1$, and we write $\overset{(1)}{x^5} + \overset{(2)}{5x^4y} + \overset{(3)}{10x^3y^2} + \overset{(4)}{10x^2y^3} + \overset{(5)}{5xy^4} + \overset{(6)}{y^5}$, which corresponds with the $5th$ power as obtained by actual multiplication.

In the same manner the coëfficients of the $6th$ power are,

$$\overset{(1)}{1}, \quad \overset{(2)}{\frac{1 \times 6}{1}}, \quad \overset{(3)}{\frac{6 \times 5}{2}}, \quad \overset{(4)}{\frac{15 \times 4}{3}}, \quad \overset{(5)}{\frac{20 \times 3}{4}}, \quad \overset{(6)}{\frac{15 \times 2}{5}}, \quad \overset{(7)}{\frac{6 \times 1}{6}}, \text{ that is,}$$

$$1, \quad 6, \quad 15, \quad 20, \quad 15, \quad 6, \quad 1,$$

and when inserted with the letters, we have,

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.$$

3. Find the $7th$ power of $x + y$.

*Ans.* $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$.

4. Find the $2nd$ power of $x + y$. (§ **62.**) *Ans.* $x^2 + 2xy + y^2$.

5. Find the $3rd$ power of $x + y$.

6. Find the $4th$ power of $x + y$.

7. Find the $8th$, $9th$, and $10th$ powers of $x + y$.

8. Find the $m^{th}$ power of $x + y$.

*Ans.* $$(x+y)^m = x^m + mx^{m-1}y + \frac{m(m-1)}{1,\ 2,}x^{m-2}y^2 + \frac{m(m-1)(m-2)}{1,\ 2,\ 3,}x^{m-3}y^3 + \frac{m(m-1)(m-2)(m-3)}{1,\ 2,\ 3,\ 4,}x^{m-4}y^4, \text{ \&c.}$$

**129.** To raise the residual monomial $(x - y)$ to any required power:

*Make the odd terms positive and the even terms negative.*

1. Find the $5th$ power of $x - y$.

*Ans.* $$\overset{(1)}{x^5} - \overset{(2)}{5x^4y} + \overset{(3)}{10x^3y^2} - \overset{(4)}{10x^2y^3} + \overset{(5)}{5xy^4} - \overset{(6)}{y^5}.$$

2. Find the $4th$ power of $x - y$.

*Ans.* $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

3. Find the $2nd$, $3rd$, $6th$, $7th$, $8th$, $9th$, and $10th$ powers of $x - y$.

4 Find the $m^{th}$ power of $(x - y)$.

*Ans.* $$(x-y)^m = x^m - mx^{m-1}y + \frac{m(m-1)}{1,\ 2,}x^{m-2}y^2 - \frac{m(m-1)(m-2)}{1,\ 2,\ 3,}x^{m-3}y^3 + \frac{m(m-1)(m-2)(m-3)}{1,\ 2,\ 3,\ 4,}x^{m-4}y^4, \text{ \&c.}$$

**130.** To raise *any* binomial to any required power:

*Consider each term of the binomial as a single expression, and proceed as in* § **128** *or* § **129**. *Then reduce the result as indicated by the signs.*

1. Find the 5*th* power of $3x + 2y$.

In the first place, $(3x + 2y)^5 = (3x)^5 + 5(3x)^4(2y) + 10(3x)^3(2y)^2 + 10(3x)^2(2y)^3 + 5(3x)(2y)^4 + (2y)^5$.

This, reduced, gives, $(3x + 2y)^5 = 243x^5 + 810x^4y + 1080x^3y^2 + 720x^2y^3 + 240xy^4 + 32y^5$.

2. Find the 3*rd* power of $2x + 3y$.

First, $(2x + 3y)^3 = (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3$, which, on reduction, gives,

$$(2x + 3y)^3 = 8x^3 + 36x^2y + 54xy^2 + 27y^3.$$

3. Find the 4*th* power of $x - 2y$.

First $(x - 2y)^4 = x^4 - 4x^3(2y) + 6x^2(2y)^2 - 4x(2y)^3 + (2y)^4$,

and this gives, $(x - 2y)^4 = x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$.

4. Find the cube of $2x^2 - 3y^3$.

First, $(2x^2 - 3y^3)^3 = (2x^2)^3 - 3(2x^2)^2(3y^3) + 3(2x^2)(3y^3)^2 - (3y^3)^3$,

and this gives, $(2x^2 - 3y^3)^3 = 8x^6 - 36x^4y^3 + 54x^2y^6 - 27y^9$.

5. Develop $(1 + x)^9$.

*Ans.* $1 + 9x + 36x^2 + 84x^3 + 126x^4 + 126x^5 + 84x^6 + 36x^7 + 9x^8 + x^9$.

6. Develop $(x + 1)^9$. *Ans.* $x^9 + 9x^8 + 36x^7$, &c.

7. Develop $(1 - x)^4$. *Ans.* $1 - 4x + 6x^2 - 4x^3 + x^4$.

8. Develop $(x - 1)^5$. *Ans.* $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$.

9. Develop $(1 - x)^6$, $(1 - x^2)^4$, $(x^2 + y^2)^5$, $(x^3 + y^3)^6$, $(2x + 2y^2)^4$, and $(3x^2 - 2y)^3$.

10. Develop $(3x - y)^4$, $(5x + 2y)^2$, $(1 + x^2)^5$, $(x^3 - 1)^5$, and $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2$.

11. Develop $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^3$. *Ans.* $x^{\frac{3}{2}} + 3xy^{\frac{1}{2}} + 3x^{\frac{1}{2}}y + y^{\frac{3}{2}}$.

12. Develop $(x^{\frac{1}{2}} - y^{\frac{1}{2}})^4$. *Ans.* $x^2 - 4x^{\frac{3}{2}}y^{\frac{1}{2}} + 6xy - 4x^{\frac{1}{2}}y^{\frac{3}{2}} + y^2$.

13. Develop $(x^{\frac{1}{3}} - y^{\frac{1}{3}})^3$. *Ans.* $x - 3x^{\frac{2}{3}}y^{\frac{1}{3}} + 3x^{\frac{1}{3}}y^{\frac{2}{3}} - y$.

**131.** The two formulae, § **128**, 8 and § **129**, 4, may be written together, giving expression to the

BINOMIAL THEOREM.

$$(x \pm y)^m = x^m \pm mx^{m-1}y + \frac{m(m-1)}{1.2}x^{m-2}y^2 \pm \frac{m(m-1)(m-2)}{1.2.3}$$
$$x^{m-3}y^3 + \frac{m(m-1)(m-2)(m-3)}{1.2.3.4}x^{m-4}y^4, \text{ \&c.}$$

If in this equation $x = 1$ and $y = 1$, we have,

$$(1 \pm 1)^m = 1 \pm m + \frac{m(m-1)}{1.2} \pm \frac{m(m-1)(m-2)}{1.2.3} +$$
$$\frac{m(m-1)(m-2)(m-3)}{1.2.3.4}. \quad (a.) \quad i.\ e.$$

The sum of the coëfficients of *any* power of $(x + y)$ is the same as 2 raised to the same power, by observing the upper signs.

If now $m = 3$, and we observe the *upper* signs, the formula becomes, $(1 + 1)^3 = (2)^3 = 1 + 3 + 3 + 1$, *i. e.* the sum of the coëfficients of $(x + y)^3 = 2^3 = 8$.

If $m = 4$, then $(1 + 1)^4 = 2^4 = 1 + 4 + 6 + 4 + 1$, which are the coëfficients of $(x + y)^4 = 16$.

If $m = 5$, then $(1 + 1)^5 = 2^5 = 1 + 5 + 10 + 10 + 5 + 1$, the coëfficients of $(x + y)^5 = 32$.

If $m = 10$, then $(1 + 1)^{10} = 2^{10} = 1 + 10 + 45 + 120 + 210 + 252 + 210 + 120 + 45 + 10 + 1$.

If $m = 3$, and we observe the *lower* signs, the formula becomes, $(1 - 1)^3 = 1 - 3 + 3 - 1$; *i. e.* the positive terms exactly cancel the negative, and in general, from formula ($a$), above, *the sum of the positive coëfficients must be exactly the same as the sum of the negative in any power of* $x - y$.

We see also, that *in calculating the binomial coëfficients, we need to actually perform the operations only half way, by taking the first half in the reverse order.*

The formula above may be used to solve *any* example.

**132**-(1). To find the cube of a trinomial, (*vide* **65**):

Take the trinomial $x + y + z$. Consider the terms $x + y$ as a single expression, and we may write the whole thus, $((x + y) + z)^3$.

This developed as already explained, gives,

$((x + y) + z)^3 = (x + y)^3 + 3\,(x + y)^2 z + 3\,(x + y)\,z^2 + z^3$,
which, reduced, gives

$(x + y + z)^3 = x^3 + 3x^2y + 3xy^2 + y^3 + 3x^2z + 6xyz + 3y^2z + 3xz^2 + 3yz^2 + z^3$.

This may be arranged thus,

$(x + y + z)^3 = x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3y^2x + 3y^2z + 3z^2x + 3z^2y + 6xyz$.

Hence, to cube a trinomial:

*Cube the three terms, and to their sum add three times the second power of each term into the first power of each of the others, and also add six times the product of all the terms.*

1. Develop $(x - y + z)^3$. *Ans.* $x^3 - y^3 + z^3 - 3x^2y + 3x^2z + 3y^2x + 3y^2z + 3z^2x - 3z^2y - 6xyz$.

2. Develop $(x - 2y + 3z)^3$. *Ans.* $x^3 - 8y^3 + 27z^3 - 4x^2y + 6x^2z + 8y^2x + 24y^2z + 18z^2x - 36z^2y - 36xyz$.

3. Develop $(x^2 + x + 1)^3$. *Ans.* $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$.

4. Develop $(x^2 - x - 1)^3$. *Ans.* $x^6 - 3x^5 + 5x^3 - 3x - 1$.

5. Develop $(x^4 - x^2 - 1)^3$. *Ans.* $x^{12} - 3x^{10} + 5x^6 - 3x^2 - 1$.

6. Develop $(x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}})^3$, $(x^{\frac{1}{3}} + x + x^{\frac{2}{3}})^3$ and $(1 - x + x^2)^3$.

7. Develop $(x - x^{\frac{1}{2}} + \frac{1}{2})^3$. *Ans.* $x^3 - 3x^{\frac{5}{2}} + \dfrac{9x^2}{2} - 4x^{\frac{3}{2}} + \dfrac{9x}{4} - \dfrac{3x^{\frac{1}{2}}}{4} + \frac{1}{8}$.

**132**-(2). Useful changes in the form of expressions.

1. $x^2 + y^2 = (x + y)^2 - 2xy$, or $x^2 + y^2 = (x - y)^2 + 2xy$.
2. $x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)$. *Vide* § **75**.
3. $x^3 - y^3 = (x - y)^3 - 3xy\,(x - y)$. *Vide* § **75**.

4. $x^4 + y^4 = (x+y)^4 - 4xy(x+y)^2 + 2x^2y^2$. *Vide* § **75**, § **76**.
5. $x^5 + y^5 = (x+y)^5 - 5xy(x+y)^3 + 5x^2y^2(x+y)$.
6. $x^5 - y^5 = (x-y)^5 + 5xy(x-y)^3 + 5x^2y^2(x-y)$.
7. $x^3 - y^3 - xy^2 + x^2y = (x-y)(x+y)^2$.
8. $x^3 + y^3 - xy^2 - x^2y = (x+y)(x-y)^2$.

**132**-(3). To find a term which will make an expression of the form $x^2 \pm 2ax$ a perfect square.

We have $(x \pm a)^2 = x^2 \pm 2ax + a^2$, where $a^2 = \frac{1}{2}(2a)$ squared.

Hence, *Square half the coëfficient of the second term, add the result to the expression, and it will be a perfect square.*

1. Make $x^2 + 2x$ a perfect square. *Ans.* $x^2 + 2x + 1$.
2. Make $x^2 + 4x$ a perfect square. *Ans.* $x^2 + 4x + 4$.
3. Make $x^2 - 6x$ a perfect square. *Ans.* $x^2 - 6x + 9$.
4. Make $x^2 - 3x$ a perfect square. *Ans.* $x^2 - 3x + \frac{9}{4}$.
5. Make $x^2 + 2px$ a perfect square. *Ans.* $x^2 + 2px + p^2$.
6. Make $x^2 - \dfrac{2bn^2x}{n^2 - m^2}$ a perfect square.

*Ans.* $x^2 - \dfrac{2bn^2x}{n^2 - m^2} + \dfrac{b^2n^4}{(n^2 - m^2)^2}$.

## LOGARITHMS.

**133.** A *logarithm* is a number expressing the power to which a given number is to be raised to produce another given number.

**134.** The number to be raised is called the *base* of the system of logarithms to which it gives rise.

**135.** In the equations,

$3^0=1$, $3^1=3$, $3^2=9$, $3^3=27$, $3^4=81$, $3^5=243$, $3^6=729$, &c.,
*three* is the *base*, and 0, 1, 2, 3, 4, 5, 6, are respectively the logarithms of 1, 3, 9, 27, 81, 243, 729.

**136.** The *base* of the Common System is 10, from which we readily form this table.

| | | | | | |
|---|---|---|---|---|---|
| $10^0 =$ | 1 | $10^4 =$ | 10000 | $10^8 =$ | 100000000 |
| $10^1 =$ | 10 | $10^5 =$ | 100000 | $10^9 =$ | 1000000000 |
| $10^2 =$ | 100 | $10^6 =$ | 1000000 | $10^{10} =$ | 10000000000 |
| $10^3 =$ | 1000 | $10^7 =$ | 10000000 | $10^{11} =$ | 100000000000 &c. |

Since $a^0 = 1$, the logarithm of 1 in *any* system is 0.

**137.** Any one of these equations is expressed generally by $\quad a^x = M$

And any other by $\quad a^y = N$

And by multiplying we have $\quad a^{x+y} = M \times N$. That is,

*The logarithm of the product of two numbers is equal to the sum of their logarithms.*

Thus, by the table above, the logarithm of 10000 is 4,
and the logarithm of 100000 is 5,
and we find the logarithm of the product of these numbers, viz: 1000000000 to be 9,
which is $5 + 4$.

**138.** Again, $\quad a^x = M$
and $\quad a^y = N$

By dividing we have $\quad a^{x-y} = \frac{M}{N}$. That is,

*The logarithm of the quotient of two numbers is equal to the difference of their logarithms.*

Thus, the logarithm of 100000000000 is 11.
The logarithm of 100000000 is 8.
And the logarithm of quotient, viz: 1000 is 3,
which is $11 - 8$.

**139**-(1). By § **126**, ex. 18, the $m^{th}$ power of the equation,

$$a^x = M$$

is $\quad a^{mx} = M^m$. That is,

*The logarithm of the $m^{th}$ power of a number is $m$ times the logarithm of the number.*

Thus, the logarithm of 100 is 2,

and the logarithm of the 5*th* power of 100, } viz: 10000000000 is 10,

which is $5 \times 2$

**139**-(2). By § 126, ex. 20, the $m^{th}$ root of the equation,

$$a^x = M$$

is $$a^{\frac{x}{m}} = \sqrt[m]{M}.$$ That is,

*The logarithm of the $m^{th}$ root of a number is the logarithm of the number divided by m.*

Thus, the logarithm of 10000000000 is 10,

and the logarithm of the 5*th* root of 10000000000, } viz: 100 is 2,

which is $10 \div 5$.

**140.** By examining the table § **136**, we see that,

The logarithms of numbers between 1 and 10 must be greater than 0 and less than 1.

The logarithms of numbers between 10 and 100 must be greater than 1 and less than 2.

The logarithms of numbers between 100 and 1000 must be greater than 2 and less than 3, &c.

The following table illustrates this, where the decimals are carried to 6 places.

| | | | |
|---|---|---|---|
| $10^3 = 1$ | $10^{.698970} = 5$ | $10^{1.477121} = 30$ | $10^{4.845098} = 70000$ |
| $10^{.301030} = 2$ | $10^{.778151} = 6$ | $10^{2.698970} = 500$ | $10^{5.301030} = 200000$ |
| $10^{.477121} = 3$ | $10^{.903090} = 8$ | $10^{2.954243} = 900$ | $10^{6.903090} = 8000000$ |
| $10^{.602060} = 4$ | $10^{1.301030} = 20$ | $10^{3.602060} = 4000$ | $10^{5.778151} = 600000$ |

The integral part of the logarithm is called the *characteristic*, and in case the number whose logarithm is in question is *without decimals*,

(*a*) The characteristic is always *less by one* than the number of figures composing it. But if the number has a *decimal* connected with it, then,

(*b*) The *characteristic* is *less by one* than the number of figures on the left of the decimal.

Thus, the characteristic of 2.5 is 0; of 25.67 it is 1; of 477.3 it is 2, &c.

**141.** By § 68, Example 4, we may write the following equations:

$$\frac{1}{10}=10^{-1}=\ .1 \quad \frac{1}{10^4}=10^{-4}=\ .0001 \quad \frac{1}{10^7}=10^{-7}=\ .0000001$$

$$\frac{1}{10^2}=10^{-2}=\ .01 \quad \frac{1}{10^5}=10^{-5}=\ .00001 \quad \frac{1}{10^8}=10^{-8}=\ .00000001$$

$$\frac{1}{10^3}=10^{-3}=.001 \quad \frac{1}{10^6}=10^{-6}=\ 000001 \quad \frac{1}{10^9}=10^{-9}=.000000001$$

Hence,

(*c*) The characteristic of the logarithm of a decimal is a *negative* number, and is always *greater by one* than the number of cyphers at the beginning of the decimal. Thus,

The characteristic of .3 is —1; of .053 it is —2; of .00057 it is —4.

To save space the sign is usually written *above* the figure, thus, $\overline{3}.602060$.

The decimal part of a logarithm is sometimes called the *mantissa*, and is always *positive*.

**142.** The equations of § **140** might be continued, and a complete table be formed, including numbers as high as we might choose to go, but such an arrangement would occupy far too much space. We may omit the *base*, 10, write the numbers in a column, and the logarithms to the right. Thus,

TABLE OF LOGARITHMS FROM 1 TO 100.

| N. | Log. | N. | Log. | N. | Log. | N. | Log. |
|---|---|---|---|---|---|---|---|
| 1 | 0.000000 | 26 | 1.414973 | 51 | 1.707570 | 76 | 1.880814 |
| 2 | 0.301030 | 27 | 1.431364 | 52 | 1.716003 | 77 | 1.886491 |
| 3 | 0.477121 | 28 | 1.447158 | 53 | 1.724276 | 78 | 1.892095 |
| 4 | 0.602060 | 29 | 1.462398 | 54 | 1.732394 | 79 | 1.897627 |
| 5 | 0.698970 | 30 | 1.477121 | 55 | 1.740363 | 80 | 1.903090 |
| 6 | 0.778151 | 31 | 1.491362 | 56 | 1.748188 | 81 | 1.908485 |
| 7 | 0.845098 | 32 | 1.505150 | 57 | 1.755875 | 82 | 1.913814 |
| 8 | 0.903090 | 33 | 1.518514 | 58 | 1.763428 | 83 | 1.919078 |
| 9 | 0.954243 | 34 | 1.531479 | 59 | 1.770852 | 84 | 1.924279 |
| 10 | 1.000000 | 35 | 1.544068 | 60 | 1.778151 | 85 | 1.929419 |
| 11 | 1.041393 | 36 | 1.556303 | 61 | 1.785330 | 86 | 1.934498 |
| 12 | 1.079181 | 37 | 1.568202 | 62 | 1.792392 | 87 | 1.939519 |
| 13 | 1.113943 | 38 | 1.579784 | 63 | 1.799341 | 88 | 1.944483 |
| 14 | 1.146128 | 39 | 1.591065 | 64 | 1.806180 | 89 | 1.949390 |
| 15 | 1.176091 | 40 | 1.602060 | 65 | 1.812913 | 90 | 1.954243 |
| 16 | 1.204120 | 41 | 1.612784 | 66 | 1.819544 | 91 | 1.959041 |
| 17 | 1.230449 | 42 | 1.623249 | 67 | 1.826075 | 92 | 1.963788 |
| 18 | 1.255273 | 43 | 1.633468 | 68 | 1.832509 | 93 | 1.968483 |
| 19 | 1.278754 | 44 | 1.643453 | 69 | 1.838849 | 94 | 1.973128 |
| 20 | 1.301030 | 45 | 1.653213 | 70 | 1.845098 | 95 | 1.977724 |
| 21 | 1.322219 | 46 | 1.662758 | 71 | 1.851258 | 96 | 1.982271 |
| 22 | 1.342423 | 47 | 1.672098 | 72 | 1.857333 | 97 | 1.986772 |
| 23 | 1.361728 | 48 | 1.681241 | 73 | 1.863323 | 98 | 1.991226 |
| 24 | 1.380211 | 49 | 1.690196 | 74 | 1.869232 | 99 | 1.995635 |
| 25 | 1.397940 | 50 | 1.698970 | 75 | 1.875061 | 100 | 2.000000 |

**143.**

EXAMPLES.

1. Multiply 2 by 28. Always take the following form.

Logarithm of 2 = 0.301030

Logarithm of 28 = 1.447158

Logarithm of 56 = 1.748188

We add the logarithms of 2 and 28, and find 1.748188, for which we must look in the table. We find it opposite 56.

2. Multiply 11 by 8.

| | | |
|---|---|---|
| Logarithm of | 11 = | 1.041393 |
| Logarithm of | 8 = | 0.903090 |
| Logarithm of | 88 = | 1.944483 |

3. Multiply 2.5 by 3. (*Vide* § **140** *b.*)

| | | |
|---|---|---|
| Logarithm of | 2.5 = | 0.397940 |
| Logarithm of | 3 = | 0.477121 |
| Logarithm of | 7.5 = | 0.875061 |

4. Multiply .4 by .0023. (*Vide* § **141** *c.*)

| | | |
|---|---|---|
| Logarithm of | .4 = | $\bar{1}.602060$ |
| Logarithm of | .0023 = | $\bar{3}.361728$ |
| Logarithm of | .00092 = | $\bar{4}.963788$ |

5. Multiply 17 by .0005.

| | | |
|---|---|---|
| Logarithm of | 17 = | 1.230449 |
| Logarithm of | .0005 = | $\bar{4}.698970$ |
| Logarithm of | .0085 = | $\bar{3}.929419$ |

6. Multiply 3 by 7, .8 by 12, .045 by .02, .07 by 1.3, &c.

**144.** Before giving other examples we will explain the use of the table at the end of the book, which contains the logarithms of all numbers between 1 and 10000. The characteristic is omitted, as it may be easily supplied by the rules above, marked (*a*), (*b*), (*c*).

(1.) If the number consists of *three figures* with cyphers prefixed or added:

> *Find these figures in the column marked* N. *Opposite the number in the next column is the mantissa of the logarithm, to which prefix the characteristic, by* (*a*), (*b*), *or* (*c*).

Since the two left-hand figures of the mantissa remain the same for several consecutive logarithms, they are printed but

*once,* in the column marked 0 at the top. These figures belong to the four figures in all the columns. Thus,

The logarithm of 364 is 2.561101. The logarithm of 365 is 2.562293. The logarithm of 739 = 2.868644.

Find the logarithms of 201, 453, 510, 620, 729, 841, 934, and 999.

The logarithm of 3.64 is 0.561101. The logarithm of .00365 is $\overline{3}$.562293. The logarithm of 73900 = 4.868644.

Find the logarithms of 28.1, 3.65, .453, .0267, .00384, 765000, and 320.

(2.) If the number consists of *four figures* with cyphers prefixed as decimals, or annexed to make up a *whole number:*

> *Find the first three figures of the number in the column marked* N. *Opposite to these in the column marked with a fourth figure at the top are four figures of the mantissa, to which prefix the two left-hand figures of the first column, and to the mantissa thus completed prefix the characteristic by* (*a*), (*b*), *or* (*c*). Thus,

The logarithm of 3171 is 3.501196. The logarithm of 3172 = 3.501333. The logarithm of 3173 = 3.501470.

Find the logarithms of 3845, 4443, 4552, 6854, 7921, and 9999.

If *dots* are observed in passing to the column marked with the fourth figure at the top, then *the two figures of the first column must be taken from the line below.* Thus,

The logarithm of 3166 = 3.500511. The logarithm of 5014 = 3.700184. The logarithm of 57.59 = 1.760347.

Find the logarithms of 5564, 537.6, 53.76, .5376, and .001234.

(3.) If the number consist of more than four figures:

> *Find the logarithms of the first four figures as in* (2).
>
> *Take the figures in the column marked* D *at the top, found in the same line with the number, and multiply them by the*

*remaining figures of the number given, pointing off from the right as many figures as are found in the multiplier.*

*Add the figures remaining at the left to the right of the logarithm already found, and the sum will be the logarithm of the given number, after prefixing the proper characteristic by (a), (b), or (c).*

1. Find the logarithm of 246891.

First, the logarithm of 246800 = 5.392345.

The number in the column marked D on the line of the number 2468 is 176, which multiplied by 91 gives 160.16. We now *add* the integral part of the product to the right-hand figures of the mantissa already found. Thus,

| | | | |
|---|---|---|---|
| Logarithm of | 246800 | = | 5.392345 |
| 176 × .91 | | = | 160 |
| Which gives Logarithm of | 246891 | = | 5.392505 |

2. In the same way find the logarithm of 6789532.

| | | | |
|---|---|---|---|
| Logarithm of | 6789000 | = | 6.831806 |
| 64 × .532 | | = | .34 |
| Logarithm of | 6789532 | = | 6.831840 |

3. Find the logarithm of 12.347.

| | | | |
|---|---|---|---|
| Logarithm of | 12.340 | = | 1.091315 |
| 351 × .7 | | = | 246 |
| Logarithm of | 12.347 | = | 1.091561 |

We add 246 because 245.7 is nearer 246 than 245. So always when the first figure in the decimal is greater than 5.

4. Verify the following.

| | | | |
|---|---|---|---|
| Logarithm of | 67895 | = | 4.831838 |
| Logarithm of | 68707 | = | 4.837001 |
| Logarithm of | 47.306 | = | 1.674916 |
| Logarithm of | 432.156 | = | 2.635640 |

Logarithm of .000432156 = $\overline{4}$.635640
Logarithm of 78.9102 = 1.897133
Logarithm of 4.32195 = 0.635679
Logarithm of .015364 =
Logarithm of 123456 =
Logarithm of .023967 =
Logarithm of .111122 =
Logarithm of .999999 =

**145.** To find the number corresponding to a given logarithm:

By ($a$) and ($b$) if the characteristic be 0 or a *positive number;* The number of figures on the left of the decimal in the required number must be *one greater than is indicated by the characteristic.*

By ($c$). If the characteristic be *negative*, the required number is a decimal fraction, having the number of cyphers between the decimal point and the first significant figure *less by one than is indicated by the characteristic of the given logarithm.*

**146.** (1.) If the mantissa of the logarithm can be exactly found:

*Find the mantissa in the table and take out the corresponding number. Point off as directed in* § **145.**

1. Find the number corresponding to logarithm 2.928396.

*Ans.* 848.

2. Find the number corresponding to logarithm 3.928396.

*Ans.* 8480.

3. Find the number corresponding to logarithm $\overline{1}$.928396.

*Ans.* .848.

4. Find the number corresponding to logarithm 3.962606.

*Ans.* 9175.

5. Find the number corresponding to logarithm $\overline{3}$.970114.

*Ans.* .009335.

(2.) If the mantissa of the logarithm cannot be exactly found:

*Take from the table the next less logarithm and the number corresponding to it.*

*Subtract this next less logarithm from the given logarithm, and divide the remainder by the number in the column marked* D, *found in the same line.*

*Annex the quotient to the number already taken out, and point off as directed in* § **145.**

1. Find the number whose logarithm is 3.123456.

*Form of the Operation.*

Logarithm of 1328.78 = 3.123456
Logarithm of 1328 = 3.123198

328)258(.78

The next less logarithm tabulated is 3.123198, and the number corresponding is 1328, which we write opposite 3.123198. Next subtract the logarithms, and we have 258, which we divide by 328, found in the column D. Annex the quotient **.78** to 1328, and it is the *required number*, viz: 1328.78.

2. Find the number whose logarithm is 1.894325.

*Operation.*

Logarithm of 78.40164 = 1.894325 = given logarithm.
1.894316 = next less logarithm.

No. in column D = 55)9000(.164 = quotient.

Why is the last figure in the quotient 4 and not 3?

3. Find the number whose logarithm is $\bar{1}.910360$.

*Operation.*

Logarithm of .813504 = $\bar{1}.910360$
58 = $\left\{\begin{array}{l}\text{last two figures of next} \\ \text{less logarithm.}\end{array}\right.$

No. in column D = 53)200(.04 = quotient.

4. Find the number whose logarithm is $\bar{2}.750360$.

*Operation.*

Logarithm of .05628078 = $\overline{2}$.750360 = given logarithm.
54 = next less logarithm.

No. in column D 77) 6000 (.078 = quotient.

5. Find the number whose logarithm is $\overline{4}$.700446.

*Ans.* .0005017023.

6. Find the number whose logarithm is 2.698971.

*Ans.* 500.0011.

7. Find the number whose logarithm is 3.602061.

*Ans.* 4000.009.

8. Find the number whose logarithm is 2.650020.

*Ans.* 446.704.

## MULTIPLICATION BY LOGARITHMS.

**147.**

1. Multiply 24.6 by 25.3. (*Vide* § **137.**)

*Operation.*

| | | | |
|---|---|---|---|
| Logarithm of | 24.6 | = | 1.390935 |
| Logarithm of | 25.3 | = | 1.403121 |
| Logarithm of | 622.38 | = | 2.794056 |

2. Multiply 52.74 by 27.

*Operation.*

| | | | |
|---|---|---|---|
| Logarithm of | 52.74 | = | 1.722140 |
| Logarithm of | 27 | = | 1.431364 |
| Logarithm of | 1423.98 | = | 3.153504 |

3. What is the product of 12 × 34.12 × .0056 × 5.671 × .8123 × .004 × 23.46?

*Operation.*

| | | | |
|---|---|---|---|
| Logarithm of | 12 | = | 1.079181 |
| Logarithm of | 34.12 | = | 1.533009 |
| Logarithm of | .0056 | = | $\bar{3}$.748188 |
| Logarithm of | 5.671 | = | 0.753660 |
| Logarithm of | .8123 | = | $\bar{1}$.909716 |
| Logarithm of | .004 | = | $\bar{3}$.602060 |
| Logarithm of | 23.46 | = | $\bar{1}$.370328 |
| Log'm of | .009911568 | = | $\bar{3}$.996142 |

4. Multiply 23.14 by 5.062. *Ans.* 117.1347.

5. Multiply 2.581926 by 3.457291. *Ans.* 8.92648.

## DIVISION BY LOGARITHMS.

**148.**
1. Divide 24163 by 4567. (*Vide* § **138.**)

*Operation.*

| | | | |
|---|---|---|---|
| Logarithm of | 24163 | = | 4.383151 |
| Logarithm of | 4567 | = | 3.659631 |
| Logarithm of | 5.29078 | = | 0.723520 |

2. Divide 2 by 3456.

*Operation.*

| | | | |
|---|---|---|---|
| Logarithm of | 2 | = | 0.301030 |
| Logarithm of | 3456 | = | 3.538574 |
| *Ans.* | .000578704 | = | $\bar{4}$.762456 |

3. Divide 1 by 256.

*Operation.*

| | | | |
|---|---|---|---|
| Logarithm of | 1 | = | 0.000000 |
| Logarithm of | 256 | = | 2.408240 |
| *Ans.* | .00390625 | = | $\bar{3}$.591760 |

4. What is the value of .8697 ÷ 98.65? *Ans.* .008816.

5. Divide 29.76 by 6254. *Ans.* .00476.

### ARITHMETICAL COMPLEMENT.

**149.** The arithmetical complement of a logarithm is the difference between *ten* and the logarithm. Thus,

The arithmetical complement of 3.602060 is $10 - 3.602060 = 6.397940$.

The arithmetical complement of $\overline{2}.698970$ is 11.301030; of $\overline{4}.477121$ it is 13.522879.

**150.** To find the arithmetical complement of a logarithm:

*Take the left-hand figure from* 9, *and proceed towards the right, taking each figure from* 9 *till the last significant figure is reached, which must be taken from* 10.

Let $x$ = any logarithm,

and $y$ = any other logarithm less than $x$,

and $c$ = the arithmetical complement of $y$.

By definition above $10 - y = c$ or $-y = c - 10$.

Therefore $x - y = x + c - 10$.

From which we see that,

*The difference between two logarithms is found by adding to the first logarithm the arithmetical complement of the other, and diminishing the sum by* 10.

1. Divide 24163 by 4567.

*Operation.*

| | | |
|---|---|---|
| Logarithm of 24163 | = 4.383151 | |
| Arithmetical comp. of logarithm of 4567 | = 6.340369 | (*Vide* §**148**, ex. 1.) |
| *Ans.* 5.29078 | = 0.723520 | = sum by rejecting 10. |

2. Divide .7438 by 12.9476.

*Operation.*

| | | |
|---|---|---|
| Logarithm of .7438 | = $\overline{1}.871456$ | |
| Logarithm of 12.9476 | = 8.887811 | arithmetical complement. |
| *Ans.* 0.057447 | = $\overline{2}.759267$ | = sum by rejecting 10. |

3. What is the value of $\frac{48 \times .75 \times 72 \times .0625}{.027 \times 120}$?

*Operation.*

| | | | |
|---|---|---|---|
| Logarithm of | 48 | | $= 1.681241$ |
| Logarithm of | .75 | | $= \bar{1}.875061$ |
| Logarithm of | 72 | | $= 1.857332$ |
| Logarithm of | .0625 | | $= \bar{2}.795880$ |
| Logarithm of | .027 | arith. comp. | $= 11.568636$ |
| Logarithm of | 120 | arith. comp. | $= 7.920819$ |
| *Ans.* | 50 | | $= 1.698969 = \begin{cases}\text{sum after re-}\\\text{jecting 2 } tens.\end{cases}$ |

4. Find the values of $\frac{6.832}{.0362}$, $\frac{.00634}{62.18}$, $\frac{3642}{23.68}$, and $\frac{.657}{.0793}$

## INVOLUTION BY LOGARITHMS

**151.**

1. What is the square of 2.5? (*Vide* § **139.**)

*Operation.*

$$\begin{array}{lrl} \text{Logarithm of} & 2.5 = & 0.397940 \\ & & \underline{\phantom{0.39794}2} \\ \textit{Ans.} & 6.25 = & 0.795880 \end{array}$$

2. What is the cube of 32.16?

$$\begin{array}{lrl} \text{Logarithm of} & 32.16 = & 1.507316 \\ & & \underline{\phantom{1.50731}3} \\ \textit{Ans.} & 33261.9 = & 4.521948 \end{array}$$

3. Find the square of 6.05987. *Ans.* 36.72203.

4. Find the 5*th* power of 2.97643. *Ans.* 233.6031.

5. Find the 7*th* power of 1.09684. *Ans.* 1.909864.

## EXTRACTION OF ROOTS BY LOGARITHMS.

**152.**

1\. Find the square root of 256. (*Vide* § **139.**)

*Operation*

Logarithm of 256 = 2.408240

Logarithm of 16 = 1.204120 = $\frac{1}{2}$ the logarithm of 256.

2\. Find the square root of 2.

*Operation.*

Logarithm of 2 = 0.301030

Logarithm of 1.41421 = 0.150515 = $\frac{1}{2}$ the logarithm of 2.

3\. Find the cube root of 2.

*Operation*

Logarithm of 2 = 0.301030

Logarithm of 1.2599 = 0.100343 = $\frac{1}{3}$ the logarithm of 2.

4\. Find the *4th* root of 2.

*Operation.*

Logarithm of 2 = 0.301030

Logarithm of 1.1892 = 0.075257$\frac{1}{2}$ = $\frac{1}{4}$ the logarithm of 2.

5\. Find the *5th* root of 7.0825.

*Operation.*

Logarithm of 7.0825 = 0.850187

Logarithm of 1.47923 = 0.170037 = $\frac{1}{5}$ the logar'm of 7.0825.

6\. Find the cube root of .023.

*Operation.*

Logarithm of .023 = $\overline{2}$.361728

= $\overline{3}$+1.361728

Logarithm of .28438 = $\overline{1}$.453909 = $\frac{1}{3}$ the logarithm of .023.

Here since the characteristic $\overline{2}$ is negative, and the mantissa .361728 *positive*, we cannot divide by 3 as it stands. The characteristic must be so modified as to be exactly divisible by 3. Now $\overline{2} = \overline{3} + 1$, and we may write the logarithm thus, $\overline{3} + 1$ .361728, which is divisible by 3.

7. Find the $5th$ root of .0621.

*Operation.*

$$\begin{aligned}\text{Logarithm of}\quad .0621 &= \bar{2}.793092\\ &= \bar{5}+3.793092\\ &= \overline{10}+8.793092\\ &= \overline{15}+13.793092\end{aligned}$$

Logarithm of $.573612 = \bar{1}.758618 = \frac{1}{5}$ the logarithm of .0621.

Here $\bar{2} = \bar{5}+3 = \overline{10}+8 = \overline{15}+13$, &c., either of which is exactly divisible by 5, and gives the quotient $\bar{1}.758618$.

8. What is the $25th$ root of 2531000000? *Ans.* 2.37756.

9. Find the value of $2^{\frac{16}{17}}$

*Operation.*

$$\begin{array}{lr}\text{Logarithm of}\quad 2 = & 0.301030\\ & 16\\ \hline 17\,) & 4.816480\\ \hline\end{array}$$

Logarithm of $1.92009 = 0.283322 = \frac{16}{17}$ of the logarithm of 2.

10. Find the $100th$ root of 5. *Ans.* 1.0162.

11. Find the cube root of 2.987635. *Ans.* 1.440265.

12. Find the value of $\left(\frac{21}{373}\right)^{\frac{2}{3}}$. *Ans.* .146895.

13. Find the value of $\left(\frac{112}{1728}\right)^{\frac{3}{5}}$. *Ans.* 1936444.

14. Find the value of $\frac{1}{7} \times (\frac{5}{8})^{\frac{1}{2}} \times .012 \times (\frac{7}{11})^{\frac{1}{3}}$. *Ans.* .0011657.

15. Find the value of $\dfrac{\frac{1}{9} \times (\frac{11}{21})^{\frac{1}{2}} \times .03 \times (15\frac{1}{5})^{\frac{1}{3}}}{7\frac{1}{3} \times (12\frac{1}{5})^{\frac{1}{3}} \times .19 \times (17\frac{1}{8})^{\frac{1}{4}}}$

*Ans.* .300916.

**153.** Since the method originally pursued in calculating the mantissa of logarithms is easily understood, we will insert an exposition of it. If we write the two series,

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1*st*, | 0 | 1 | 2 | 3 | 4 | 5 | &c. |
| 2*nd*, | 1 | 10 | 100 | 1000 | 10000 | 100000 | &c. |

it is at once seen that *logarithms are a series of numbers in arithmetical progression corresponding to a series in geometrical progression.* To compute the logarithm of any intermediate number,

*Find the geometrical mean of the two terms of the second series between which the given number is found.*

*Find the arithmetical mean of the two corresponding terms of the first series.*

Again, *Find the geometrical mean between this new term and the term nearest the given number.*

*Find the corresponding arithmetical mean in the first series. Continue this operation till the given number becomes the geometrical mean, when the corresponding arithmetical mean will be the required logarithm.*

EXAMPLE.

Suppose it be required to find the logarithm of 9.

The geometrical mean between 10 and 1 is $\sqrt{10 \times 1}$

$= \sqrt{10} = 3.1622777$

The arithmetical mean between 0 and 1 is $\frac{1+0}{2} = \frac{1}{2} = .5$

Therefore the logarithm of 3.1622777 = .5.

Again, The next geometrical mean is $\sqrt{3.1622777 \times 10} = 5.6234132$

The arithmetical mean between 1 and .5 is $\frac{1+.5}{2} = .75$

Therefore the logarithm of 5.6234132 = .75.

*3dly.* The next geometrical mean is $\sqrt{10 \times 5.6235132} = 7.4989422$

The arithmetical mean is $\frac{1+.75}{2} = .875$

Therefore the logarithm of 7.4989422 is .875.

*4thly.* The next geometrical mean is $\sqrt{10 \times 7.4989422} = 8.6596431$

The arithmetical mean is $\frac{1.875}{2} = .9375$

Therefore the logarithm of 8.6586431 is .9375.

5*thly*. The next geometrical mean is $\sqrt{10 \times 8.6596431} = 9.3057204$

The arithmetical mean is $\dfrac{1.9375}{2} = .96875$

Therefore the logarithm of 9.3057209 is .96875.

6*thly*. The next geometrical mean is

$$\sqrt{8.6596431 \times 9.3057204} = 8.9768713$$

The arithmetical mean is $\dfrac{.9375 + .96875}{2} = .953125$

Therefore the logarithm of 8.9768713 is .953125.

Proceeding in this manner, after 25 extractions, we should find that the logarithm of 8.9999998 is .9542425, and that is the logarithm of 9, near enough for all practical purposes.

In this manner Mr. Henry Briggs found the logarithms of all the prime numbers from 1 to 20,000, and from 90,000 to 101,000, carrying the decimal part to 14 places. The student will be glad to learn that in the light of modern analysis all this labor would be lost.

## EVOLUTION AND TREATMENT OF RADICALS.

**154.** 1. *Evolution investigates the method of finding any root of a quantity.*

2. *A surd is a quantity which requires a radical sign, or index, to exactly express it.*

3. *A rational quantity requires no radical sign to express it.* Thus,

3 is a rational quantity, but $\sqrt{3}$ is a surd,
$x$ is rational, but $\sqrt[3]{x}$ is a surd.

4. *The coëfficient of a surd is the quantity prefixed to it.* Thus,

$5x^{\frac{1}{2}}$, where 5 is the coëfficient of $x^{\frac{1}{2}}$ or $\sqrt{x}$.

5. *A rational quantity may have the form of a surd.* Thus,

$$2 = \sqrt{4} = 4^{\frac{1}{2}}.$$

6. A surd is in its simplest form when, from the nature of the root required, the part under the radical sign, or fractional index, is the *smallest possible whole number.*

CASE I.

**155.** To place a surd involving an integral number in its simplest form.

*Separate the number into two factors, such that the root of one of them may be exactly taken. Take this root for a coëfficient of the other factor affected by the proper sign.*

1. Find the simplest form of $\sqrt{8}$.

$$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}. \quad \textit{Ans.}$$

2. Find the simplest form of $\sqrt[3]{16}$.

$$\sqrt[3]{16} = \sqrt[3]{8 \times 2} = \sqrt[3]{8} \times \sqrt[3]{2} = 2\sqrt[3]{2}. \quad \textit{Ans.}$$

3. Simplify $\sqrt{18}$, $\sqrt{32}$, $\sqrt{50}$, $\sqrt{72}$, $\sqrt{128}$, $\sqrt{162}$, $\sqrt{200}$, $\sqrt{242}$, and $\sqrt{288}$. *Ans.* $3\sqrt{2}$, $4\sqrt{2}$, $5\sqrt{2}$, $6\sqrt{2}$, $8\sqrt{2}$, $9\sqrt{2}$, &c.

4. Simplify $\sqrt{12}$, $\sqrt{27}$, $\sqrt{48}$, $\sqrt{75}$, $\sqrt{108}$, $\sqrt{147}$, and $\sqrt{192}$.
*Ans.* $2\sqrt{3}$, $3\sqrt{3}$, $4\sqrt{3}$, &c.

5. Simplify $\sqrt{20}$, $\sqrt{28}$, $\sqrt{44}$, $\sqrt{117}$, $\sqrt{68}$.
*Ans.* $2\sqrt{5}$, $2\sqrt{7}$, $2\sqrt{11}$, $3\sqrt{13}$, and $2\sqrt{17}$.

6. Simplify $\sqrt{76}$, $\sqrt{54}$, $\sqrt{2048}$, $\sqrt{84}$, $\sqrt{189}$, $\sqrt{480}$, and $\sqrt{338}$.

7. Simplify $\sqrt{392}$, $\sqrt{675}$, $\sqrt{1280}$, $\sqrt{2023}$, $\sqrt{3564}$, and $\sqrt{4693}$.

8. Simplify $\sqrt[3]{54}$, $\sqrt[3]{128}$, $\sqrt[3]{250}$, $\sqrt[3]{432}$, $\sqrt[3]{686}$, and $\sqrt[3]{1024}$.
*Ans.* $3\sqrt[3]{2}$, $4\sqrt[3]{2}$, $5\sqrt[3]{2}$, $6\sqrt[3]{2}$, $7\sqrt[3]{2}$, and $8\sqrt[3]{2}$.

9. Simplify $\sqrt[3]{81}$, $\sqrt[3]{135}$, $\sqrt[3]{189}$, $\sqrt[3]{297}$, and $\sqrt[3]{351}$.

10. Simplify $\sqrt[3]{320}$, $\sqrt[3]{448}$, $\sqrt[3]{704}$, $\sqrt[3]{1125}$, and $\sqrt[3]{2376}$.

11. What is the square root of 8?
*Ans.* $2\sqrt{2} = 2 \times 1.41421 = 2.82842$.

12. What is the square root of 18?

*Ans.* $3\sqrt{2} = 3 \times 1.41421 = 4.24463$.

13. What is the square root of 32, 50, 72, 128, 162, and 200?

14. Find the numerical value of all the preceding problems by the tables.

CASE II.

**156.** To place a surd involving a vulgar fraction in its simplest form.

*Multiply the numerator and denominator of the fraction by such a number as will render the denominator a perfect square, cube, &c. as the case may require.*

Simplify the numerator by Case I., and write the required root of the denominator under the coëfficient.

1. Find the simplest form of $\sqrt{\frac{4}{27}}$.

$$\sqrt{\tfrac{4}{27}} = \sqrt{\tfrac{4 \times 3}{81}} = \sqrt{\tfrac{4}{81}} \times \sqrt{3} = \tfrac{2}{9}\sqrt{3}$$

2. Find the simplest form of $\sqrt{\frac{2}{3}}$.

$$\sqrt{\tfrac{2}{3}} = \sqrt{\tfrac{6}{9}} = \sqrt{\tfrac{1}{9} \times 6} = \sqrt{\tfrac{1}{9}} \times \sqrt{6} = \tfrac{1}{3}\sqrt{6}$$

3. Simplify $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{5}}$, $\sqrt{\frac{1}{6}}$, $\sqrt{\frac{1}{7}}$, $\sqrt{\frac{1}{11}}$, &c.

*Ans.* $\frac{1}{3}\sqrt{3}$, $\frac{1}{5}\sqrt{5}$, $\frac{1}{6}\sqrt{6}$, $\frac{1}{7}\sqrt{7}$, $\frac{1}{11}\sqrt{11}$, &c

4. Simplify $\sqrt{\frac{1}{8}}$, $\sqrt{\frac{3}{32}}$, $\sqrt{\frac{64}{125}}$, $\sqrt{\frac{49}{48}}$, $\sqrt{\frac{27}{50}}$, and $\sqrt{\frac{63}{20}}$.

*Ans.* $\frac{1}{4}\sqrt{2}$, $\frac{1}{8}\sqrt{6}$, $\frac{8}{25}\sqrt{5}$, $\frac{7}{12}\sqrt{3}$, $\frac{3}{10}\sqrt{6}$, and $\frac{3}{10}\sqrt{35}$.

5. Simplify $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{3}}$, $\sqrt[3]{\frac{1}{4}}$, $\sqrt[3]{\frac{1}{7}}$, and $\sqrt[3]{\frac{1}{9}}$.

*Ans.* $\frac{1}{2}\sqrt{2}$, $\frac{1}{2}\sqrt[3]{4}$, $\frac{1}{3}\sqrt[3]{9}$, $\frac{1}{4}\sqrt[3]{2}$, $\frac{1}{7}\sqrt[3]{49}$, and $\frac{1}{3}\sqrt[3]{3}$.

6. Simplify $\sqrt{\frac{3}{5}}$, $\sqrt{\frac{7}{8}}$, $\sqrt{\frac{12}{25}}$, $\sqrt{\frac{11}{18}}$, $\sqrt{\frac{5}{9}}$, $\sqrt{\frac{8}{11}}$, $\sqrt{\frac{4}{15}}$, and $\sqrt{\frac{3}{8}}$.

*Ans.* $\frac{1}{5}\sqrt{15}$, $\frac{1}{4}\sqrt{14}$, $\frac{2}{5}\sqrt{3}$, $\frac{1}{6}\sqrt{22}$, $\frac{1}{3}\sqrt{5}$, $\frac{2}{11}\sqrt{22}$, $\frac{3}{15}\sqrt{15}$, and $\frac{1}{4}\sqrt{6}$.

7. Simplify $9\sqrt{\frac{16}{27}}$, $6\sqrt{\frac{25}{12}}$, $5\sqrt{\frac{9}{10}}$, $10\sqrt{\frac{3}{50}}$, and $7\sqrt{\frac{3}{28}}$.

*Ans.* $4\sqrt{3}$, $5\sqrt{3}$, $\frac{3}{2}\sqrt{10}$, $\sqrt{6}$, and $\frac{1}{2}\sqrt{21}$.

8. What is the square root of $\frac{1}{2}$?

*Ans.* $\sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2} = \frac{1}{2}$ of $1.41421 = .7071$.

9. What is the square root of $\frac{1}{8}$? *Ans.* $\frac{1}{4}\sqrt{2} = .35355$.

10. What is the cube root of $\frac{1}{2}$?

*Ans.* $\sqrt[3]{\frac{1}{2}} = \frac{1}{2}\sqrt[3]{4} = \frac{1}{2}$ of $1.5874 = .7937$.

11. What is the value of $9\sqrt{\frac{16}{27}}$?

*Ans.* $4\sqrt{3} = 4 \times 1.73204 = 6.92816$.

12. What is the value of $\sqrt{\frac{4}{99}}$? *Ans.* $\frac{2}{99}\sqrt{99} = .20107$.

13. What is the value of $\sqrt{\frac{28}{35}}$? *Ans.* $\frac{2}{35}\sqrt{7} = .15118$.

14. What is the value of $\sqrt{\frac{81}{5}}$? *Ans.* $\frac{9}{5}\sqrt{5} = 4.02492$.

### CASE III.

**157.** To find the root of a *positive algebraic monomial.*

Take the required root of the coëfficient, and divide the exponent of each letter by the index of the root.

1. Find the square root of $49x^4y^6$. *Ans.* $7x^2y^3$.
2. What is the value of $\sqrt{289x^2y^{10}}$. *Ans.* $17xy^5$.
3. What are the values of $\sqrt{361x^2y^4}$, $\sqrt{441x^2y^{20}}$, and $\sqrt{256x^4y^8}$?
4. What are the values of $\sqrt{324x^2y^{12}}$, $\sqrt{400x^4y^{14}}$, and $\sqrt{484x^8y^{12}}$?
5. What are the values of $\sqrt[3]{8x^3y^6}$, $\sqrt{64x^6y^{12}}$, and $\sqrt[4]{16x^4y^8}$?
6. What are the values of $\sqrt[5]{243x^5y^{10}}$, $\sqrt[10]{1024x^{10}y^{30}}$, and $\sqrt[6]{729x^{12}y^{18}}$?

### CASE IV.

**158.** To find the root of a *negative algebraic monomial.*

(1.) $+a \times +a = +a^2$, and $-a \times -a = +a^2$, $\therefore$

*The even root of a negative quantity is impossible.*

(2.) $-a \times -a \times -a = -a^3 \therefore \sqrt[3]{-a^3} = -a$. Hence,

*The odd root of a negative quantity is negative.*

(3.) $\pm a \times \pm a = a^2 \therefore \sqrt{a^2} = \pm a$. Hence,

*The even root of a positive quantity is positive or negative.*

When the double sign $\pm$ occurs two or more times in the same equation, the upper signs must not be confounded with the lower. Thus, $\pm a \mp b = \pm c$ means $+a - b = c$, or $-a + b = -c$.

1. Find the cube root of $-8x^{12}y^{24}$. *Ans.* $-2x^4y^8$.

2. What is the value of $\sqrt[5]{-32x^5y^{10}}$? *Ans.* $-2xy^2$.

3. What are the values of $\sqrt[4]{16x^4y^8}$, $\sqrt[3]{-27x^6y^3}$, and $\sqrt{169x^4y^8}$?

4. What are the values of $\sqrt[10]{x^{10}y^{20}}$, $\sqrt[5]{32x^5y^{10}}$, and $\sqrt{9a^2b^2}$?

5. What are the values of $\sqrt[4]{-16a^4}$, $\sqrt[7]{-2187x^{14}y^{21}}$, and $\sqrt{529x^2y^4}$?

6. What are the values of $\sqrt{x^4y^8}$, $\sqrt[3]{64x^6y^{12}}$, and $\sqrt[3]{-216x^6y^{12}}$?

## CASE V.

**159.** To simplify algebraic monomials whose root cannot be exactly taken.

*Simplify the numerical part by* I. or II.

Divide each exponent by the index of the root to be taken, and write the letters with the *quotient* for an exponent on the outside of the radical sign, and the letters with the *remainder* for an exponent *under* the radical sign.

### EXAMPLES.

1. What is the simplest form of $\sqrt{18x^4y^4}$? *Ans.* $3x^2y^2\sqrt{2}$.

2. What is the simplest form of $\sqrt[3]{54x^6y^5}$? *Ans.* $3xy\sqrt[3]{2y^2}$.

3. Simplify $\sqrt[4]{32x^4y^8}$, $\sqrt[6]{192x^8y^5}$, $\sqrt{\frac{2}{3}x^4y^5}$, and $\sqrt[3]{\frac{1}{2}x^3y^4}$.

4. Simplify $\sqrt{\frac{5}{9}x^4y^6}$, $\sqrt{\frac{9}{10}x^4y^2}$, $\sqrt{\frac{300}{5}x^6y^8}$, and $\sqrt[3]{16x^3y^4}$.

5. Simplify $\sqrt[3]{128x^{12}y^8}$, $\sqrt[5]{64x^{10}y^{15}}$, $\sqrt{\frac{1}{2}x^3y^4}$, and $\sqrt{\frac{4}{5}x^4y^9}$.

6. Simplify $\sqrt[4]{\frac{2}{5}x^4y^8}$, $\sqrt{\frac{2}{5}x^8y^{12}}$, $5\sqrt{\frac{2}{5}x^5y^7}$, and $15\sqrt{8x^4y^5}$.

7. Simplify $\sqrt{44x^4y^5}$, $\sqrt{75x^5y^7}$, $\sqrt{8x^4y^{11}}$, and $7\sqrt{28x^5y^6}$.

8. Simplify $\sqrt{50x^7y^8}$, $\sqrt{200x^4y^5}$, $\sqrt{243xy^{10}}$, and $\sqrt{\frac{1}{2}x^9y^{10}}$.

9. Simplify $\sqrt{\frac{1}{4}x^8y^9}$, $\sqrt{\frac{2}{7}x^4y^5}$, $\sqrt{\frac{3}{8}xy^{15}}$, and $\sqrt{xyz^{15}}$.

## CASE VI.

**160.** To add radical quantities.

*Simplify each expression by* Case V.

If the radical parts are then the same, *add the coëfficients and prefix the sum to the common radical.*

If the radical parts are not the same, *unite the quantities by the proper sign.*

EXAMPLES.

1. Add $\sqrt{27x^4y^2}$, $\sqrt{48x^4y^2}$, $\sqrt{75x^4y^2}$ and $\sqrt{192x^4y^2}$.

*Operation.*

$$\begin{aligned} \sqrt{27x^4y^2} &= 3x^2y\sqrt{3} \\ \sqrt{48x^4y^2} &= 4x^2y\sqrt{3} \\ \sqrt{75x^4y^2} &= 5x^2y\sqrt{3} \\ \sqrt{192x^4y^2} &= 8x^2y\sqrt{3} \\ \hline & \quad 20x^2y\sqrt{3} = \text{the sum.} \end{aligned}$$

2. Add together $\sqrt[3]{54x^6y^5}$, and $\sqrt[3]{128x^{12}y^8}$.

*Operation.*

$$\begin{aligned} \sqrt[3]{54x^6y^5} &= 3xy\sqrt[3]{2y^2} \\ \sqrt[3]{128x^{12}y^8} &= 4x^4y^2\sqrt[3]{2y^2} \\ \hline & \quad (3xy+4x^4y^2)\sqrt[3]{2y^2} = \text{the sum.} \end{aligned}$$

3. Add together $\sqrt{320x^8y^4}$, $\sqrt{180x^8y^4}$, $\sqrt{245x^8y^4}$, and $\sqrt{20x^8y^4}$.

*Ans.* $25x^4y^2\sqrt{5}$.

4. Add together $\sqrt{28x^4y^2}$, $\sqrt{63x^4y^2}$, and $\sqrt{112x^4y^2}$.

*Ans.* $9x^2y\sqrt{7}$.

5. Add together $\sqrt{99x^3y^5}$, $\sqrt{275x^3y^5}$, $\sqrt{176x^3y^5}$, and $\sqrt{44x^3y^5}$.
6. Add together $\sqrt{\frac{5}{7}x^2y^4}$, $\sqrt{\frac{7}{5}x^2y^4}$, $\sqrt{140x^2y^4}$, and $\sqrt{315x^2y^4}$.
7. Add together $\sqrt{\frac{3}{8}x^2y^6}$, $\sqrt{\frac{8}{3}x^2y^6}$, $\sqrt{\frac{1}{6}x^2y^6}$, and $\sqrt{\frac{27}{8}x^2y^6}$.
8. Add together $\sqrt{\frac{3}{7}x^2y^5}$, $\sqrt{\frac{7}{3}x^2y^5}$, $\frac{1}{5}\sqrt{84x^2y^5}$, and $\frac{1}{9}\sqrt{189x^2y^5}$.
9. Add together $\sqrt{\frac{4}{5}x^2y^4}$, $\sqrt{\frac{5}{4}x^2y^4}$, $\sqrt{\frac{40}{18}x^2y^4}$, and $\sqrt{20x^2y^4}$.
10. Add together $\sqrt[3]{24x^3y^6}$, $\sqrt[3]{192x^3y^6}$, $\sqrt[3]{81x^3y^6}$, and $\sqrt[3]{375x^3y^6}$.
11. Add together $\sqrt[4]{32x^4y^8}$, $\sqrt[4]{162x^4y^8}$, $\sqrt[4]{512x^4y^8}$, and $\sqrt[4]{1250x^4y^8}$.
12. Add together $\sqrt[3]{\frac{1}{2}x^3y^7}$, $\sqrt[3]{\frac{1}{16}x^3y^6}$, $\sqrt[3]{\frac{4}{27}x^3y^6}$, and $\sqrt[3]{\frac{1}{54}x^3y^6}$.

13. Add together $\sqrt{\frac{1}{2}x^2y^4}$, $\sqrt{\frac{1}{3}x^2y^4}$, and $\sqrt{\frac{1}{4}x^2y^5}$.

14. Add together $\sqrt{\frac{1}{7}x^2y^4}$, $\sqrt{\frac{7}{4}x^4y^6}$, and $\sqrt{\frac{2}{2}\frac{1}{7}x^4y^4}$.

15. Add together $\sqrt[3]{24x^3y^6}$, $\sqrt[3]{32x^3y^6}$, and $\sqrt[3]{40x^3y^6}$.

16. Add together $\sqrt{20x^4y^6}$, $\sqrt{28x^4y^6}$, $\sqrt{36x^4y^6}$, and $\sqrt{44x^4y^6}$.

17. Add together $\sqrt{\frac{1}{3}x^2y^2z^2}$, $\sqrt{\frac{1}{5}x^2y^2z^2}$, and $\sqrt{\frac{1}{7}x^2y^2z^2}$.

18. Add together $\sqrt{49x^2y^4}$, $\sqrt{64x^2y^4}$, $\sqrt{81x^2y^4}$, and $\sqrt{8x^2y^4}$.

19. Add together $\sqrt{2x^2 + 4xy + 2y^2}$, and $\sqrt{2x^2 - 4xy + 2y^2}$.

*Ans.* $2x\sqrt{2}$.

CASE VII.

**161.** To subtract radical quantities.

*Simplify each expression by* Case V.

If the radical parts are then the same, *subtract the coëfficients and prefix the difference to the common radical.*

If the radical parts are not the same, *express the difference by the proper sign.*

EXAMPLES.

1. From $\sqrt{18x^2y^2}$ take $\sqrt{8x^2y^2}$.

*Operation.*

$$\begin{aligned}\sqrt{18x^2y^2} &= 3xy\sqrt{2}\\ \sqrt{8x^2y^2} &= \underline{2xy\sqrt{2}}\\ &\phantom{=}\ xy\sqrt{2} = \text{the difference.}\end{aligned}$$

2. From $\sqrt{\frac{2}{4}x^4y^2}$ take $\sqrt{\frac{2}{9}x^4y^2}$. *Ans.* $\frac{1}{6}x^2y\sqrt{2}$.

3. From $\sqrt{63x^8y^4}$ take $\sqrt{28x^8y^4}$. *Ans.* $x^4y^2\sqrt{7}$.

4. From $\sqrt{80x^4y^2}$ take $\sqrt{20x^4y^2}$.

5. From $\sqrt{\frac{5}{4}x^2y^6}$ take $\sqrt{\frac{4}{5}x^2y^6}$.

6. From $\sqrt{\frac{6}{5}x^2y^4}$ take $\sqrt{\frac{5}{6}x^2y^4}$.

7. From $\sqrt{\frac{14}{32}x^4y^6}$ take $\sqrt{\frac{48}{21}x^4y^6}$.

8. From $\sqrt{3x^6y^6}$ take $\sqrt{\frac{1}{3}x^6y^6}$.

9. From $\sqrt[3]{27x^3y^9}$ take $\sqrt[3]{8x^3y^9}$.

10. From $\sqrt[3]{192x^{12}y^{15}}$ take $\sqrt[3]{81x^{12}y^{15}}$.

11. From $\sqrt[4]{16x^4y^8}$ take $\sqrt[4]{x^4y^8}$. 12. From $\sqrt[5]{32x^5y^{11}}$ take $\sqrt[5]{x^5y^{11}}$.

13. From $\sqrt[6]{64x^6y^{13}}$ take $\sqrt[6]{x^6y^{12}}$.

14. From $\sqrt{\frac{27}{174}x^4y^2}$ take $\sqrt{\frac{1}{58}x^4y^2}$.

15. From $\sqrt{\frac{1}{18}x^4y^5}$ take $\sqrt{\frac{2}{49}x^4y^5}$. 16. From $2\sqrt{\frac{3}{4}}$ take $3\sqrt{\frac{1}{3}}$.

17 From $\sqrt{\frac{2}{3}}$ take $\sqrt{\frac{2}{27}}$.

18. From $\sqrt{2x^2+4xy+2y^2}$ take $\sqrt{2x^2-4xy+4y^2}$. *Ans.* $2y\sqrt{2}$.

## CASE VIII.

**162.** To multiply radical quantities; multiply the coëfficients and also the radicals.

*Simplify the result by* Case V.

### EXAMPLES.

1. Multiply $7\sqrt{5x^2y^3}$ by $4\sqrt{\frac{2}{5}x^4y}$.

*Operation.*

$$\begin{array}{l} 7\sqrt{5x^2y^3} \\ 4\sqrt{\frac{2}{5}x^4y} \\ \hline 28\sqrt{2x^6y^4} = 28x^3y^2\sqrt{2} = \text{product.} \end{array}$$

2. Multiply $\frac{5}{3}\sqrt{\frac{3}{8}x^2y}$ by $\frac{1}{5}\sqrt{\frac{2}{5}yz^2}$.

*Operation.*

$$\begin{array}{l} \frac{5}{3}\sqrt{\frac{3}{8}x^2y} \\ \frac{1}{5}\sqrt{\frac{2}{5}yz^2} \\ \hline \frac{1}{3}\sqrt{\frac{3}{20}x^2y^2z^2} = \frac{1}{3}xyz\sqrt{\frac{15}{100}} = \frac{1}{30}xyz\sqrt{15} = \text{product.} \end{array}$$

3. Multiply $4\sqrt[3]{2}$ by $2\sqrt[3]{4}$. *Ans.* 16.

4. Multiply $\sqrt{x}$ by $\sqrt{x}$. *Ans.* $x$.

5. Multiply $3\sqrt{5xy}$ by $4\sqrt{20x}$. *Ans.* $120x\sqrt{y}$.

6. Multiply $7\sqrt{3x^2y^3}$ by $8\sqrt{16\frac{1}{3}xy^2}$. *Ans.* $392xy^2\sqrt{xy}$.

7. Multiply $5\sqrt{\frac{1}{3}xy^4}$ by $\frac{1}{5}\sqrt{27xy}$. *Ans.* $3xy^2\sqrt{y}$.

8. Multiply $5\sqrt{\frac{2}{5}xy^4}$ by $\frac{1}{10}\sqrt{40xy^2}$. *Ans.* $2xy^3$.

9. Multiply $\sqrt{x^2+1}$ by $\sqrt{x^2+1}$. *Ans.* $x^2+1$.

10. Multiply $\sqrt{x+1}$ by $\sqrt{x-1}$. *Ans.* $\sqrt{x^2-1}$.

11. Multiply $\sqrt{x^2+a^2}$ by $\sqrt{x^2-a^2}$.

12. Multiply $\sqrt{x+\sqrt{y}}$ by $\sqrt{x-\sqrt{y}}$.

13. Multiply $\sqrt{3+\sqrt{2}}$ by $\sqrt{3-\sqrt{2}}$.

14. Multiply $\sqrt{7+\sqrt{24}}$ by $\sqrt{7-\sqrt{24}}$.

15. Multiply $\sqrt{5+\sqrt{2}}$ by $\sqrt{5+\sqrt{2}}$.

16. Multiply $\sqrt{5x}+\sqrt{4y}$ by $\sqrt{5x}-\sqrt{4y}$.

17. Multiply $\sqrt{7}+\sqrt{3}$ by $\sqrt{7}-\sqrt{3}$.

18. Multiply $\sqrt{15}+\sqrt{2}$ by $\sqrt{15}+\sqrt{2}$.

19. Multiply $\sqrt{2x}+\sqrt{3y}$ by $\sqrt{2x}+\sqrt{3y}$.

20. Multiply $\sqrt{2}+\sqrt{18}$ by $\sqrt{2}+\sqrt{18}$.

21. Multiply $\sqrt{5x}+\sqrt{8y}$ by $\sqrt{5x}+\sqrt{8y}$.

22. Multiply $4\sqrt{2x}-3\sqrt{2y}$ by $5\sqrt{8x}+7\sqrt{18y}$.

23. Multiply $5\sqrt{2x}+3\sqrt{18y}$ by $3\sqrt{18x}-5\sqrt{2y}$.

24. Multiply $\sqrt{x}+\sqrt{y}$ by $\sqrt{x}-\sqrt{y}$.

25. Multiply $\dfrac{\sqrt{x}+7}{\sqrt{x}-7}$ by $\dfrac{\sqrt{x}+4}{\sqrt{x}-4}$.

26. Multiply $\dfrac{\sqrt{x}+5}{\sqrt{x}+7}$ by $\dfrac{\sqrt{x}-5}{\sqrt{x}-7}$.

27. Multiply $x+\sqrt{xy}+y$ by $x-\sqrt{xy}+y$. *Ans.* $x^2+xy+y^2$.

28. Multiply $-a+\sqrt{a^2+b}$ by $-a-\sqrt{a^2+b}$. *Ans.* $-b$.

29. Multiply $a+\sqrt{a^2+b}$ by $a-\sqrt{a^2+b}$. *Ans.* $-b$.

30. Multiply $6+\sqrt{36+2}$ by $6-\sqrt{36-2}$. *Ans.* $-2$

31. Multiply $\sqrt{x+y}+\sqrt{x-y}$ by $\sqrt{x+y}-\sqrt{x-y}$.
*Ans.* $2y$.

32. Multiply $\sqrt{x}+3$ by $\sqrt{x}-3$, also $\sqrt{x}+2$ by $\sqrt{x}-2$, and $\sqrt{x}+4$ by $\sqrt{x}-4$.

### CASE IX.

**163.** To divide radical quantities.

*Simplify each expression by* Case V.

Prefix the quotient of the coëfficients to the quotient of the radical parts, and, if necessary, again simplify for the final result.

### EXAMPLES.

1. Divide $\frac{1}{8}\sqrt{28x^3y^5}$ by $\frac{2}{3}\sqrt{63xy^2}$.

*Operation.*

$$\begin{aligned} \tfrac{1}{8}\sqrt{28x^3y^5} &= \tfrac{1}{4}xy^2\sqrt{7xy} \\ \tfrac{2}{3}\sqrt{63xy^2} &= \underline{\quad 2y\sqrt{7x}\quad} \\ &\quad \tfrac{1}{8}xy\sqrt{y} = \text{quotient.} \end{aligned}$$

2. Divide $\sqrt{20x^2y}$ by $\sqrt{8xy}$. *Ans.* $\frac{1}{2}\sqrt{10x}$.

3. Divide $\sqrt{x^2} + \sqrt{xy} + \sqrt{xy^2}$ by $\sqrt{x}$. *Ans.* $\sqrt{x} + \sqrt{y} + y$.

4. Divide $\sqrt{28x^3y^5} + \sqrt{63x^3y^5} + \sqrt{112x^3y^5}$ by $\sqrt{7xy}$.
*Ans.* $9xy^2$.

5. Divide $\frac{1}{7}\sqrt{2x^2y^2}$ by $\frac{1}{7}\sqrt{\frac{1}{4}xy}$. *Ans.* $2\sqrt{2xy}$.

6. Divide $\sqrt{16 + \sqrt{4}}$ by $4\sqrt{\frac{1}{2}}$. *Ans.* $\frac{3}{2}$.

7. Divide $\sqrt{19 - \sqrt{9}}$ by 2. *Ans.* 2.

8. Divide $x^2 + xy + y^2$ by $x + \sqrt{xy} + y$. *Ans.* $x - \sqrt{xy} + y$.

9. Find the values of $\frac{\sqrt{54}}{\sqrt{6}}$, $\frac{8\sqrt{50}}{4\sqrt{2}}$, $\frac{12\sqrt{28}}{3\sqrt{7}}$, $\frac{15\sqrt{378}}{5\sqrt{6}}$, $\frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{3}}}$, and $\frac{\sqrt{\frac{3}{4}}}{\sqrt{\frac{1}{3}}}$.
*Ans.* 3, 10, 8, $9\sqrt{7}$, $\frac{1}{2}\sqrt{6}$, and $\frac{3}{2}$.

10. Find the values of $\frac{\frac{2}{3}\sqrt{18}}{\frac{1}{2}\sqrt{2}}$, $\frac{\frac{3}{5}\sqrt{\frac{1}{3}}}{\frac{1}{2}\sqrt{\frac{3}{5}}}$, $\frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{\frac{1}{4}\sqrt{\frac{1}{2}}}$, and $\frac{\sqrt{2} + 3\sqrt{\frac{1}{2}}}{\frac{1}{2}\sqrt{\frac{1}{2}}}$.
*Ans.* 4, $\frac{2}{5}\sqrt{5}$, 2, and 10.

CASE X.

**164.** To reduce a fraction, whose denominator is a binomial containing radical quantities, to an equivalent fraction without radicals in the denominator.

*Multiply both the numerator and denominator of the fraction by the denominator with one sign changed.*

EXAMPLES.

1. Reduce the fraction $\frac{3}{\sqrt{5}+\sqrt{2}}$ to a fraction having no radicals in the denominator.

*Operation.*

$$\frac{3}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{3\sqrt{5}-3\sqrt{2}}{3} = \sqrt{5}-\sqrt{2} = .82185.$$

2. Find the value of $\frac{2}{\sqrt{3}+\sqrt{5}}$.

*Ans.* $\frac{2\sqrt{3}-2\sqrt{5}}{-2} = \sqrt{5}-\sqrt{3} = .50401.$

3. Find the value of $\frac{3}{8+\sqrt{2}}$. *Ans.* .31866.

4. Find the values of $\frac{5}{7-\sqrt{40}}$, $\frac{8}{\sqrt{3}-\sqrt{7}}$, $\frac{5}{9-\sqrt{8}}$, and $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

5. Find the values of $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$, $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$, and $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$.

6. Find the values of $\frac{3}{\sqrt{5}-\sqrt{2}}$, and $\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}$.

*Ans.* 3.65028, 2.80588.

7. Find the values of $\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}$, and $\frac{2\sqrt{11}-3\sqrt{13}}{2\sqrt{11}+3\sqrt{13}}$.

8. Given $\frac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}}$, to free the denominator of radicals.

*Ans.* $\frac{2a^2 - x^2 - 2a\sqrt{a^2 - x^2}}{x^2}$

9. Given $\frac{\sqrt{x} + \sqrt{x - a}}{\sqrt{x} - \sqrt{x - a}}$, to free the denominator of radicals.

*Ans.* $\frac{2x - a + 2\sqrt{x^2 - ax}}{a}$.

CASE XI.

**165.** To find the square root of a polynomial.

1. If necessary, arrange the polynomial with reference to a given letter, and place the square root of the first term to the right of the polynomial, for the first term of the root. Square this term, and subtract it from the polynomial.

2. Double this first term of the root, and place it on the left of the remainder for a part of the divisor; divide the first term of the remainder by this double of the root, and place the quotient in the root as the second term, *and also* at the right of the divisor.

3. Multiply the *whole* divisor by the second term of the root, and subtract the product from the remainder.

4. Double the whole of the root, and write the result to the left of the remainder, as a part of the divisor; divide the first term of the remainder by the first term of the partial divisor, and place the result as the third term of the root, *and also* at the right of the partial divisor.

5. Multiply the whole divisor by the third term of the root, and subtract the product from the last remainder.

6. In a similar manner find other terms.

EXAMPLES.

1. What is the square root of $x^4 + 4x^3 + 6x^2 + 4x + 1$?

*Operation.*

$$
\begin{array}{r|l|l}
 & x^4 + 4x^3 + 6x^2 + 4x + 1 & \underline{x^2 + 2x + 1.}\ \textit{Ans.} \\
 & x^4 & \\ \hline
2x^2 + 2x & 4x^3 + 6x^2 + 4x + 1 & \\
 & 4x^3 + 4x^2 & \\ \hline
2x^2 + 4x + 1 & 2x^2 + 4x + 1 & \\
 & 2x^2 + 4x + 1 & \\ \hline
\end{array}
$$

2. What is the square root of $x^4 - 2x^3 + 3x^2 - 2x + 1$?

*Operation.*

$$
\begin{array}{r|l|l}
 & x^4 - 2x^3 + 3x^2 - 2x + 1 & \underline{x^2 - x + 1} \\
 & x^4 & \\ \hline
2x^2 - x & -2x^3 + 3x^2 - 2x + 1 & \\
 & -2x^3 + x^2 & \\ \hline
2x^2 - 2x + 1 & 2x^2 - 2x + 1 & \\
 & 2x^2 - 2x + 1 & \\ \hline
\end{array}
$$

3. What is the square root of $x^2 + 2x^3 + 3x^4 + 2x^5 + x^6$?

*Ans.* $x(1 + x + x^2)$.

4. What is the square root of $x^2 - 2x^3 + 3x^4 - 2x^5 + x^6$?

*Ans.* $x(1 - x + x^2)$.

5. What is the square root of $x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz$?

*Ans.* $x + 2y + 3z$.

6. What is the square root of $x^4 + 3x^2 + 2x^3 - 2x + 1$?

*Ans.* $x^2 + x - 1$.

7. What is the square root of $x^2 + 2xy + y^2 + 2x + 2y + 1$?

*Ans.* $x + y + 1$.

8. What is the square root of $x^6 - 2x^5 - x^4 + 4x^3 - x^2 - 2x + 1$?

*Ans.* $x^3 - x^2 - x + 1$.

9. What is the square root of $1 - x^2$?

*Ans.* $1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16}$, &c.

MISCELLANEOUS.

**166.**

1. Complete the square of $x^2 + 2px$, and take the square root. (*Vide* **132**, 3.) *Ans.* $\sqrt{x^2 + 2px + p^2} = x + p$.

2. Complete the square of $x^2 + 2x$, and take the square root.

*Ans.* $x + 1$.

3. Complete the square of $x^2 - 3x$, and take the square root.

*Ans.* $x - \frac{3}{2}$.

4. Complete the square of $x^2 - \frac{2b\,n^2x}{n^2 - m^2}$, and take the square root.

*Ans.* $x - \frac{bn^2}{n^2 - m^2}$.

CASE XII.

IMAGINARY QUANTITIES.

**167.** An *imaginary quantity* is an indicated *even* root of a *negative quantity*. Its general form is—

$$\pm A\sqrt{-1}$$

where $A$ is either rational or radical.

The rules for multiplying imaginary quantities depend upon the fundamental principle that *the square root of a quantity multiplied by its square root produces the quantity itself.* Thus,

$$\sqrt{-1} \times \sqrt{-1} = -1.$$

From this we have the following table of equations.

$$\sqrt{-x} \times \sqrt{-y} = \sqrt{x}\sqrt{-1} \times \sqrt{y}\sqrt{-1} = \sqrt{xy} \times -1 = -\sqrt{xy}. \quad 1.$$

$$-\sqrt{-x} \times -\sqrt{-y} = -\sqrt{x}\sqrt{-1} \times -\sqrt{y}\sqrt{-1} = \sqrt{xy} \times -1 = -\sqrt{xy}. \quad 2.$$

$$\sqrt{-x} \times -\sqrt{-y} = \sqrt{x}\sqrt{-1} \times -\sqrt{y}\sqrt{-1} = -\sqrt{xy} \times -1 = \sqrt{xy}. \quad 3.$$

$$-\sqrt{-x} \times \sqrt{-y} = -\sqrt{x}\sqrt{-1} \times \sqrt{y}\sqrt{-1} = -\sqrt{xy} \times -1 = \sqrt{xy}. \quad 4.$$

$$\sqrt{-x} \times \sqrt{-x} = \sqrt{x}\sqrt{-1} \times \sqrt{x}\sqrt{-1} = x \times -1 = -x \quad . \ 5.$$
$$-\sqrt{-x} \times -\sqrt{-x} = -\sqrt{x}\sqrt{-1} \times -\sqrt{x}\sqrt{-1} = x \times -1 = -x \quad . \ 6.$$
$$\sqrt{-x} \times -\sqrt{-x} = \sqrt{x}\sqrt{-1} \times -\sqrt{x}\sqrt{-1} = -x \times -1 = x \quad . \ 7.$$
$$-\sqrt{-x} \times \sqrt{-x} = -\sqrt{x}\sqrt{-1} \times \sqrt{x}\sqrt{-1} = -x \times -1 = x \quad . \ 8.$$

Hence, *like* signs, on the outside of two imaginary quantities, produce *minus*, and *unlike* signs, *plus*, and the product is *not imaginary*.

It must be remembered that this apparent exception to the common rule of multiplication, applies only when the quantities to be multiplied are *both* imaginary; for

$$\sqrt{3} \times \sqrt{-5} = \sqrt{-15} = \sqrt{15}\ \sqrt{-1}, \text{ and}$$
$$\sqrt{3} \times \sqrt{-3} = \sqrt{-9} = 3\sqrt{-1}, \text{ \&c.}$$

EXAMPLES.

1. Multiply $-1 + \sqrt{-3}$ by $-1 + \sqrt{-3}$.

*Operation.*

$$\begin{array}{r} -1 + \sqrt{-3} \\ -1 + \sqrt{-3} \\ \hline 1 - \sqrt{-3} \\ -\sqrt{-3} - 3 \\ \hline -2 - 2\sqrt{-3}. \end{array} \ \textit{Ans.}$$

2. Multiply $-2 - 2\sqrt{-3}$ by $-1 + \sqrt{-3}$

*Operation.*

$$\begin{array}{r} -2 - 2\sqrt{-3} \\ -1 + \sqrt{-3} \\ \hline 2 + 2\sqrt{-3} \\ -2\sqrt{-3} + 6 \\ \hline 8 = 2 + 6. \end{array} \ \textit{Ans.}$$

By comparing these two examples we see that $(-1+\sqrt{-3})^3=8$.

We should also find $(-1-\sqrt{-3})^3=8$.

3. Multiply $\sqrt{-2}+\sqrt{-1}+\sqrt{-5}$ by $\sqrt{-2}-\sqrt{-1}-\sqrt{-5}$.

*Operation.*

$$\begin{array}{llll}
\sqrt{-2}+\sqrt{-1}+\sqrt{-5} & & & \\
\sqrt{-2}-\sqrt{-1}-\sqrt{-5} & & & \\
\hline
-2-\sqrt{2} & -\sqrt{10} & & \\
+1+\sqrt{2} & +\sqrt{10} & +\sqrt{5} & \\
+5 & & +\sqrt{5} & \\
\hline
4+2\sqrt{5}=\textit{Ans.} & & &
\end{array}$$

4. Multiply $5\sqrt{-1}+3\sqrt{-28}+2\sqrt{-7}$ by $6\sqrt{-4}-3\sqrt{-9}+5\sqrt{-7}$.

*Operation.*

$$5\sqrt{-1}+3\sqrt{-28}+2\sqrt{-7}=5\sqrt{-1}+8\sqrt{-7}\ldots\text{by VI.}$$

$$6\sqrt{-4}-3\sqrt{-9}+5\sqrt{-7}=3\sqrt{-1}+5\sqrt{-7}\ldots\text{by VII}$$

$$\begin{array}{rrr}
-15 & -\;24\sqrt{7} & \\
 & -\;25\sqrt{7} & -280 \\
\hline
-295 & -\;49\sqrt{7} & =\textit{Ans.}
\end{array}$$

5. Multiply $\sqrt{-x}+\sqrt{-xy}+\sqrt{-y}$ by $\sqrt{-x}-\sqrt{-xy}+\sqrt{-y}$. *Ans.* $-x+xy-y-2\sqrt{xy}$.

6. Find the value of $(\sqrt{-5}+\sqrt{-2})^2$. *Ans.* $-7-2\sqrt{10}$.

7. Find the value of $(\sqrt{-8}+\sqrt{-5})^2$. *Ans.* $-13-4\sqrt{10}$.

8. Find the value of $(\sqrt{-27}+\sqrt{-3})^2$. *Ans.* $-48$.

9. Find the value of $(\sqrt{-28}-\sqrt{-7})^2$. *Ans.* $-7$.

10. Find the value of $\sqrt{-13}-\sqrt{-5})^2$. *Ans.* $-18+2\sqrt{65}$.

11. Find the value of $(\sqrt{-1}+\sqrt{-2})(\sqrt{-1}-\sqrt{-2})$. *Ans.* 1.

12. Find the value of $(\sqrt{-3}+\sqrt{-5})(\sqrt{-3}-\sqrt{-5})$.

*Ans.* 2

13. Find the value of $(-1-\sqrt{-3})^3$.

*Ans.* $(-1-\sqrt{-3})^3=(-1)^3-3(-1)^2\sqrt{-3}+3(-1)$ $\sqrt{-3^2}-\sqrt{-3^3}=8$.

14. Find the value of $(-1+\sqrt{-3})^3$. *Ans.* 8.

15. Find the different powers of $\sqrt{-1}$.

*Ans.* $\sqrt{-1}\times\sqrt{-1}=-1 \quad = 2nd$ power of $\sqrt{-1}$.

$-1\times\sqrt{-1}=-\sqrt{-1}=3rd$ " " $\sqrt{-1}$.

$-\sqrt{-1}\times\sqrt{-1}=1 \quad =4th$ " " $\sqrt{-1}$.

$1\times\sqrt{-1}=\sqrt{-1} \quad =5th$ " " $\sqrt{-1}$.

And since the quantity and its 5th power are the same, it follows that the 2nd and 6th powers must be the same, so also must the 3rd and 7th, the 4th and 8th, &c., be alike.

16. Find the value of $(2+5\sqrt{-1})^4+(2-5\sqrt{-1})^4$.

*Ans.* 82.

17. Find the value of $(1+\sqrt{-11})^5-(1-\sqrt{-11})^5$.

*Ans.* $=992$.

18. Find the value of $\dfrac{1}{\frac{1}{2}(1+\sqrt{-1})}+\dfrac{1}{\frac{1}{2}(1-\sqrt{-1})}$. *Ans.* 2.

# CHAPTER VII.

## EQUATIONS OF THE SECOND DEGREE.

**168.** An equation of the *second degree* is one involving the *second power* of the unknown quantity.

Such equations may be *complete* or *incomplete.*

*A complete equation* is one involving both the first and second degrees of the unknown quantity. Thus, $x^2 + 2ax = q$.

*An incomplete equation* is one involving only the second degree of the unknown quantity. Thus, $2ax^2 = q$.

**169.** To find the value of $x$ in an incomplete equation.

*Proceed exactly as with simple equations of one unknown quantity, and take the square root of the final equation for the value of* $x$.

EXAMPLES.

1. Given $x^2 - 192 = \frac{x^2}{4}$, to find the values of $x$.

*Operation.*

| | |
|---|---|
| $x^2 - 192 = \frac{x^2}{4}$ | (1) = given equation. |
| $4x^2 - 768 = x^2$ | (2) = (1) $\times$ 4. |
| $3x^2 = 768$ | (3) = (2) reduced. |
| $x^2 = 256$ | (4) = (3) $\div$ 3. |
| $x = \pm 16$ | (5) = $\sqrt{(4)}$. (*Vide* **158.** 3.) |

2. Given $\frac{x^2}{3} - 8 = \frac{x^2}{9} + 10$, to find the values of $x$.

*Ans.* $x = \pm 9$.

3. Given $8 + 5x^2 = \frac{x^2}{5} + 4x^2 + 28$, to find the values of $x$.

*Ans.* $x = \pm 5$.

4. Given $\frac{x^2}{8} = -67\frac{3}{8} + \frac{3x^2}{2}$, to find $x$. *Ans.* $x = \pm 7$.

5. Given $\frac{x^2}{5} + 4 = \frac{7x^2}{3} - 15\frac{1}{5}$, to find $x$. *Ans.* $x = \pm 3$.

6. Given $\frac{3x^2 + 5}{8} - \frac{x^2 + 29}{3} = 117 - 5x^2$, to find $x$.

*Ans.* $x = \pm 5$.

7. Given $\frac{5x^2}{3} + 12 = \frac{8x^2}{7} + 37\frac{2}{3}$, to find $x$. *Ans.* $x = \pm 7$.

8. Given $\frac{x^2}{3} - 1 = \frac{4x^2}{27} + \frac{2}{3}$, to find $x$. *Ans.* $x = \pm 3$.

9. Given $\frac{x + 7}{x^2 - 7x} - \frac{x - 7}{x^2 + 7x} = \frac{7}{x^2 - 73}$, to find $x$.

*Ans.* $x = \pm 9$.

10. Given $(x^2 + 1)^2 = 25$, to find $x$.

*Operation.*

$$
\begin{aligned}
(x^2 + 1)^2 &= 25 && (1) = \text{given equation.} \\
x^2 + 1 &= \pm 5 && (2) = \sqrt{(1)}. \\
x^2 &= 4 \text{ or } -6 && (3) = (2) \text{ reduced.} \\
x &= \pm 2 \text{ or } \pm \sqrt{-6} && (4) = \sqrt{(3)}.
\end{aligned}
$$

11. Given $\frac{(x + 18)^2}{28} = \frac{4x^2}{63}$, to find $x$.

*Operation.*

$$
\begin{aligned}
\frac{(x + 18)^2}{28} &= \frac{4x^2}{63} && (1) = \text{given equation.} \\
\frac{(x + 18)^2}{4} &= \frac{4x^2}{9} && (2) = (1) \times 7. \\
\frac{x + 18}{2} &= \pm \frac{2x}{3} && (3) = \sqrt{(2)} \\
x &= 54 \text{ or } -7\tfrac{5}{7} && (4) = (3) \text{ reduced.}
\end{aligned}
$$

12. Given $\frac{(x+3)^2}{48} = \frac{4x^2}{75}$, to find $x$. *Ans.* $x = 5$ or $-\frac{15}{13}$.

13. Given $\frac{80}{5(x-7)^2} = \frac{168\frac{3}{4}}{3x^2}$, to find $x$. *Ans.* $x = 15$ or $4\frac{13}{23}$.

14. Given $\frac{x^2+3}{2x^2-13} = \frac{2x^2-13}{x^2+3}$, to find $x$. *Ans.* $x = \pm 4$.

15. Given $ax^2 = b$, to find $x$. *Ans* $x = \pm\sqrt{\frac{b}{a}} = \frac{1}{a}\sqrt{ab}$.

16. Given $x^2 + ab = 7x^2$, to find $x$. *Ans.* $x = \frac{1}{6}\sqrt{6ab}$.

17. Given $(x+a)^2 = 2ax + b$, to find $x$. *Ans.* $x = \pm\sqrt{b-a^2}$.

18. Given $\frac{(x^2+a)^2}{(x^2-b)^2} = c^2$, to find $x$.

*Ans.* $x = \pm\sqrt{\frac{a+bc}{c-1}}$ or $\sqrt{\frac{bc-a}{c+1}}$.

19. Given $\frac{x+a}{x-a} + \frac{x-a}{x+a} = \frac{10a^2}{x^2-a^2}$, to find $x$.

*Ans.* $x = \pm 2a$.

20. Given $\frac{x-a}{a} - \frac{a-2x}{x-a} = \frac{x^2+bx}{x^2-a^2}$, to find $x$.

*Ans.* $x = \pm\sqrt{ab}$.

PROBLEMS PRODUCING INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

**170.** 1. Find a number whose $\frac{5}{6}$ multiplied by its $\frac{3}{7}$ will be equal to 2520.

Let $x =$ the number; then $\frac{5x}{6} \times \frac{3x}{7} = \frac{15x^2}{42} = 2520$.

$$\therefore x = \pm 84.$$

2. Two numbers are to each other as 3 to 7, and the sum of their squares is 522. What are the numbers? Let $3x$ and $7x =$ the numbers.

The equation is $9x^2 + 49x^2 = 522$.

$\therefore x = 3$, and $3x = 9$, and $7x = 21$. *Ans.* 9 and 21.

3. Two numbers are to each other as $\frac{2}{3}$ to $\frac{3}{4}$, and the difference of their squares is 153. What are the numbers?

*Ans.* 24 and 27.

4. If 4 be added to a certain number and also subtracted, the product of the sum and difference will be 609. What is the number? *Ans.* 25.

5. If 9 be added to a certain number and also subtracted, $\frac{2}{3}$ of the product of the sum and difference will be 162. What is the number? *Ans.* 18.

6. A and B start from different points at the same time, and travel towards each other. On meeting, A has travelled 20 miles farther than B, and A would have gone B's distance in 75 hours, but B would have travelled A's distance in 108 hours. What distance had been travelled by each?

Let $x =$ B's distance; then

$x + 20 =$ A's distance $\therefore$

$\frac{x}{75} =$ A's progress per hour, and

$\frac{x+20}{108} =$ B's " " "

Now A's distance : B's distance :: A's hourly progress : B's hourly progress; or

$$x + 20 : x :: \frac{x}{75} : \frac{x+20}{108}.$$

$$\frac{(x+20)^2}{108} = \frac{x^2}{75}.$$

*Ans.* A 120, B 100 miles.

7. Two numbers are to each other as $a$ is to $b$, and the sum of their squares is $c$. What are the numbers?

The equation is $a^2x^2 + b^2x^2 = c$, whence

$$x = \frac{\sqrt{c}}{\sqrt{a^2+b^2}}; \text{ then the numbers are } \frac{a\sqrt{c}}{\sqrt{a^2+b^2}} \text{ and } \frac{b\sqrt{c}}{\sqrt{a^2+b^2}}.$$

8. Two numbers are to each other as $a$ to $b$, and the difference of their squares is $c$. What are the numbers?

$$Ans. \quad \frac{a\sqrt{c}}{\sqrt{a^2 - c^2}} \text{ and } \frac{b\sqrt{c}}{\sqrt{a^2 - c^2}}.$$

9. A man drew from a cask of wine containing $a$ gallons a certain quantity, and then filled it with water. He then drew of the mixture the same number of gallons as before, and again filled the cask with water. Having done the same a third and fourth time, he has $b$ gallons of wine left in the cask. How many gallons of wine were drawn off each time?

*Ans.* $a^{\frac{3}{4}}(a^{\frac{1}{4}} - b^{\frac{1}{4}}), a^{\frac{1}{4}} b^{\frac{1}{4}}(a^{\frac{1}{4}} - b^{\frac{1}{4}}), a^{\frac{1}{4}} b^{\frac{2}{4}}(a^{\frac{1}{4}} - b^{\frac{1}{4}}), b^{\frac{3}{4}}(a^{\frac{1}{4}} - b^{\frac{1}{4}})$.

If $a = 256$ and $b = 81$, then 64, 48, 36, and 27 are the answers.

If $a = 625$ and $b = 256$, then 125, 100, 80, and 64 are the answers.

## COMPLETE EQUATIONS OF THE SECOND DEGREE.

**171.** To solve the equation

$$x^2 + 2px = q, \qquad (1) \quad \textit{Vide} \ \S\ \mathbf{132}. \quad (3.)$$

By adding $p^2$ to both sides of this equation (Ax. II.), we have

$$x^2 + 2px + p^2 = p^2 + q \qquad (2)$$

By extracting the square root of both numbers (Ax. VI.), we have

$$x + p = \pm\sqrt{p^2 + q} \qquad (3)$$

By transposing $p$, we have

$$x = -p \pm \sqrt{p^2 + q} \qquad (4)$$

If the *upper* sign is employed, the answer is called the *first root.*

If the *lower* sign is employed, the answer is called the *second root.*

**172.** To solve the equation

$$x^2 - 2px = q. \qquad (1)$$

By taking the same steps as above, we have

$$x = p \pm \sqrt{p^2 + q}. \qquad (2)$$

**173.** To solve the equation

$$x^2 + 2px = -q. \qquad (1)$$

By taking the same steps as in § 171, we have

$$x = -p \pm \sqrt{p^2 - q}. \qquad (2)$$

**174.** To solve the equation

$$x^2 - 2px = -q. \qquad (1)$$

By taking the same steps again, we have

$$x = p \pm \sqrt{p^2 - q}. \qquad (2)$$

**175.** The four cases above solved are comprehended in the following general rule for the solution of complete equations of the second degree.

### RULE.

x *is equal to half the coëfficient of the second term taken with a contrary sign,* $\pm$ *the square root of the square of this half coëfficient united to the second member of the equation as indicated by its sign.*

### EXAMPLES.

1. Given $x^2 + 4x = 21$, to find $x$.

*Ans.* $x = -2 \pm \sqrt{4 + 21}$, $x = 3$ or $-7$.

2. Given $x^2 + 8x = 20$, to find $x$.

*Ans.* $x = -4 \pm \sqrt{16 + 20} = 2$ or $-10$.

3. Given $x^2 + 10x = 11$, to find $x$.

*Ans.* $x = -5 \pm \sqrt{25 + 11} = 1$ or $-11$.

4. Given $x^2 + 3x = 28$, to find $x$.

*Ans.* $x = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + 28} = 4$ or $-7$.

5. Given $x^2 - 4x = 21$, to find $x$.

*Ans.* $x = 2 \pm \sqrt{4 + 21} = 7$ or $-3$.

6. Given $x^2 - 8x = 20$, to find $x$.

*Ans.* $x = 4 \pm \sqrt{16 + 20} = 10$ or $-2$

7. Given $x^2 - 10x = 11$, to find $x$.

*Ans.* $x = 5 \pm \sqrt{25 + 11} = 11$ or $-1$.

8. Given $x^2 - 3x = 28$, to find $x$.

*Ans.* $x = \frac{3}{2} \pm \sqrt{\frac{9}{4} + 28} = 7$ or $-4$.

9. Given $x^2 + 6x = -8$, to find $x$.

*Ans.* $x = -3 \pm \sqrt{9 - 8} = -2$ or $-4$.

10. Given $x^2 + 8x = -15$, to find $x$.

*Ans.* $x = -4 \pm \sqrt{16 - 15} = -3$ or $-5$.

11. Given $x^2 + 10x = -16$, to find $x$.

*Ans.* $x = -5 \pm \sqrt{25 - 16} = -2$ or $-8$.

12. Given $x^2 + 5x = -6$, to find $x$.

*Ans.* $x = -\frac{5}{2} \pm \sqrt{\frac{25}{4} - 6} = -2$ or $-3$.

13. Given $x^2 - 6x = -8$, to find $x$.

*Ans.* $x = 3 \pm \sqrt{9 - 8} = 4$ or $2$.

14. Given $x^2 - 8x = -7$ to find $x$.

*Ans.* $x = 4 \pm \sqrt{16 - 7} = 7$ or $1$.

15. Given $x^2 - 11x = -28$, to find $x$.

*Ans.* $x = \frac{11}{2} \pm \sqrt{\frac{121}{4} - 28} = 7$ or $4$.

16. Given $x^2 - 15x = -56$, to find $x$.

*Ans.* $x = \frac{15}{2} \pm \sqrt{\frac{225}{4} - 56} = 8$ or $7$.

17. $x^2 + x = 6$. *Ans.* $2, -3$.

18. $x^2 + 2x = 8$. *Ans.* $2, -4$.

19. $x^2 + 3x = 18$. *Ans.* $3, -6$.

20. $x^2 + 4x = 32$. *Ans.* $4, -8$.

21. $x^2 + 5x = 50$. *Ans.* $5, -10$.

22. $x^2 + 6x = 27$. *Ans.* $3, -9$.

23. $x^2 + 13x = 68$. *Ans.* $4, -17$.

24. $x^2 + 15x = 154$. *Ans.* $7, -22$.

25. $x^2 + 20x = 125$. *Ans.* $5, -25$.

26. $x^2 + 21x = 196$. *Ans.* $7, -28$.

27. $x^2 - x = 132.$ *Ans.* $-11, +12.$

28. $x^2 - 4x = 32.$ *Ans.* $+8, -4.$

29. $x^2 - 6x = 27.$ *Ans.* $-3, +9.$

30. $x^2 - 28x = 29.$ *Ans.* $-1, +29.$

31. $x^2 - 5x = -6.$ *Ans.* $+3, +2.$

32. $x^2 - 7x = -12.$ *Ans.* $3, 4.$

33. $x^2 - 11x = -30.$ *Ans.* $5, 6.$

34. $x^2 - 16x = -63.$ *Ans.* $7, 9.$

35. $x^2 - 20x = -96.$ *Ans.* $8, 12.$

36. $x^2 - 36x = -320.$ *Ans.* $20, 16.$

37. $x^2 - 38x = -240.$ *Ans.* $8, 30.$

38. $x^2 + 5x = -6.$ *Ans.* $-2, -3.$

39. $x^2 + 14x = -45.$ *Ans.* $-5, -9.$

40. $x^2 + 8x = -15.$ *Ans.* $-5, -3.$

41. $x^2 + 10x = -21.$ *Ans.* $-7, -3.$

42. $x^2 + 14x = -48.$ *Ans.* $-8, -6.$

43. $x^2 + 20x = -36.$ *Ans.* $-18, -2.$

44. $x^2 + 16x = -63.$ *Ans.* $-9, -7.$

**176.** All the equations in §§ 171 and 174 inclusive are called *reduced equations.* Before applying the rule in § 175, the *given* equation must be brought to the form of one of the *reduced equations,* by any process thought to be most convenient; that is,

1. *All the terms involving* $x^2$ *must be united in one term, which must stand first.*

2. *All the terms involving* $x$ *must be united in one term, which must stand second.*

3. *All the remaining terms must be placed to the right of the sign of equality, united in as few terms as possible.*

4. *Divide the whole equation by the coëfficient of* $x^2$.

5. *If then* $x^2$ *has the sign* $-$, *change all the signs of the equation.*

EXAMPLES.

1. Given $-\dfrac{8x^2}{3} + 22x - 15 = \dfrac{7x^2}{3} - 28x + 30$ (1), to find $x$.

*Operation.*

$-5x^2 + 50x = 45$ (2) = (1) transposed and united.

$-x^2 + 10x = 9$ (3) = (2) ÷ 5.

$x - 10x = -9$ (4) = (3) with signs changed.

$x = 5 \pm \sqrt{25 - 9} = 9$ or 1.

2. Given $2x^2 + 8x + 7 = \dfrac{5x}{4} - \dfrac{x^2}{8} + 197$ (1), to find $x$.

$16x^2 + 64x + 56 = 10x - x^2 + 1576$ (2) = (1) cleared of fractions.

$$17x^2 + 54x = 1520. \qquad (3)$$

$$x^2 + \frac{54x}{17} = \frac{1520}{17}. \qquad (4)$$

$$x = -\frac{27}{17} \pm \sqrt{\frac{729}{289} + \frac{1520}{17}} \qquad (5)$$

$$x = -\frac{27}{17} \pm \frac{163}{17} = 8 \text{ or } -11\tfrac{3}{17}. \ (6)$$

3. Given $5x^2 + 3x = 530$, to find $x$. (1)

$$x^2 + \frac{3x}{5} = 106 \qquad (2)$$

$$x = -\frac{3}{10} \pm \sqrt{\frac{9}{100} + 106} \qquad (3)$$

$$x = 10 \text{ or } -10\tfrac{3}{5}. \qquad (4)$$

4. $3x^2 - 2x = 8$. *Ans.* 2, $-1\frac{1}{3}$.

5. $2x^2 - 7x = 72$. *Ans.* 8, $-4\frac{1}{2}$.

6. $2x - 5x^2 = -3$. *Ans.* 1, $-\frac{3}{5}$.

7. $17x - 4x^2 = 18$. *Ans.* 2, $2\frac{1}{4}$.

8. $3x - 21x^2 = -78$. *Ans.* 2, $-1\frac{6}{7}$.

9. $3x^2 + \frac{2}{3} = 3x$. *Ans.* $\frac{1}{3}$ or $\frac{2}{3}$.

10. $x^2 + 12x - 16 = 92$. *Ans.* 6 or $-18$.

11. $5x^2 + \dfrac{7x}{2} = 7x^2 - 51$. *Ans.* 6, $4\frac{1}{4}$.

12. $\dfrac{x^2}{4} - \dfrac{5x}{24} + 7 = 6\frac{23}{24}$. *Ans.* $\frac{1}{2}$, $\frac{1}{3}$.

13. $5x^2 - \dfrac{x}{2} = 78$. *Ans.* 4, $-3\frac{9}{10}$.

14. $\dfrac{10}{x} - \dfrac{14 - 2x}{x^2} = \dfrac{22}{9}$. *Ans.* 3, $\frac{21}{11}$.

15. $6x^2 - x = 92$. *Ans.* 4, $-3\frac{5}{6}$.

16. $4x - \dfrac{14 - x}{x + 1} = 14$. *Ans.* 4, $-1\frac{3}{4}$.

17. $x^2 - \dfrac{5x}{6} = -\frac{1}{6}$. *Ans.* $\frac{1}{2}$, $\frac{1}{3}$.

18. $2x^2 - 14x = 16$. *Ans.* 8, $-1$.

19. $\dfrac{x + 4}{3} - \dfrac{7 - x}{x - 3} = -1 + \dfrac{4x + 7}{9}$. *Ans.* 21, 5.

20. $3x - \dfrac{3x - 3}{x - 3} = \dfrac{3x}{2} - 3$. *Ans.* 4, $-1$.

21. $2x^2 - 30x + 3 = -x^2 + 3\frac{3}{10}x - \frac{3}{10}$. *Ans.* 11, $\frac{1}{10}$.

22. $x^2 - 3x = \dfrac{x}{4} - \dfrac{3}{4}$. *Ans.* 3, $\frac{1}{4}$.

23. $x^2 + 18x = -80$. *Ans.* $-10$, $-8$.

24. $x^2 + 5x + \frac{1}{4} = -5\frac{1}{10}x - \frac{3}{4}$. *Ans.* $-10$, $-\frac{1}{10}$.

25. $8x^2 + \dfrac{3x}{5} = -17x - 3\frac{1}{5}$. *Ans.* $-2$, $-\frac{1}{5}$.

26. $\dfrac{22 - x}{20} = \dfrac{15 - x}{x - 6}$. *Ans.* 36, 12.

27. $\dfrac{x + 3}{x} + \dfrac{7x}{x + 3} = 5\frac{3}{4}$. *Ans.* 4, 1.

28. $x + \dfrac{24}{x - 1} = 3x - 4$. *Ans.* 5, $-2$.

29. $\dfrac{x}{x+8} = \dfrac{x+3}{2x+1}$. *Ans.* 12, $-2$.

30. $\dfrac{x}{x+1} + \dfrac{x+1}{x} = 2\frac{1}{6}$. *Ans.* 2, $-3$.

31. $\dfrac{x+4}{3} - \dfrac{4x+7}{9} = \dfrac{7-x}{x-3} - 1$. *Ans.* 21, 5.

32. $\dfrac{2x-10}{8-x} - \dfrac{x+3}{x-2} = 2$. *Ans.* 7, $\frac{4}{5}$.

33. $\dfrac{1}{x-1} - \dfrac{1}{x+3} = \dfrac{1}{35}$. *Ans.* 11, $-13$.

34. Given $x^2 - 2x = 7$, to find $x$.

*Ans.* $x = 1 \pm 2\sqrt{2} = 3.82842$ or $-1.82841$.

35. Given $x^2 - 5x = 12$, to find $x$. *Ans.* $x = 6.772$ or $-1.772$.

36. Given $x^2 - 3x = 20$, to find $x$. *Ans.* $x = 6.2169$ or $-3.2169$.

37. Given $x^2 + 4x = 10$, to find $x$. *Ans.* $x = 1.7416$ or $-5.7416$.

38. Given $17x^2 - x = 21$, to find $x$

*Ans.* $x = 1.1412$ or $-1.0824$.

39. Given $13x^2 + 2x = 100$, to find $x$.

*Ans.* $x = 2.6975$ or $-2.8514$.

40. Given $x^2 - 8x = 14$, to find $x$. *Ans.* $x = 9.4772$ or $-1.4772$.

41. Given $x^2 + 8x = 8$, to find $x$. *Ans.* $x = 0.8989$ or $-8.8989$

42. Given $x^2 - 2x = -10$, to find $x$.

*Ans.* $x = 1 \pm \sqrt{1-10} = 1 \pm 3\sqrt{-1}$.

43. Given $x^2 - 20x = -104$, to find $x$.

*Ans.* $x = 10 \pm 2\sqrt{-1}$.

44. Given $x^2 - 10x = -26$, to find $x$. *Ans.* $x = 5 \pm \sqrt{-1}$.

45. Given $x^2 - 12x = -72$, to find $x$.

*Ans.* $x = 6\,(1 \pm \sqrt{-1})$.

46. Given $x^2 - 18x = -162$, to find $x$.

*Ans.* $x = 9\,(1 \pm \sqrt{-1})$.

47. Given $x + \dfrac{8}{x} = 4$, to find $x$. *Ans.* $x = 2\,(1 \pm \sqrt{-1})$.

## TRINOMIAL EQUATIONS.

**177.** A *trinomial equation* is of the form

$$x^{2n} + 2px^n = q,$$

where $n$ is any number whatever. To solve such an equation, apply the *Rule* under § 175 for the value of $x^n$, after which the value of $x$ is determined as in incomplete equations of the second degree. Thus, $x^n = p \pm \sqrt{p^2 + q}$, and $x = \sqrt[n]{p \pm \sqrt{p^2 + q}}$.

### EXAMPLES.

1. Given $x^4 + 4x^2 = 32$, to find $x$.

First, $x^2 = -2 \pm \sqrt{4 + 32} = -2 \pm 6 = 4$ or $-8$.

Second, $x = \pm 2$ or $\pm 2\sqrt{-2}$.

2. Given $x^4 - 74x^2 = -1225$, to find $x$.

*Ans.* $x = \pm 7$ or $\pm 5$.

3. Given $x^4 - 4x^2 = 9$, to find $x$. *Ans.* $x = \pm 3$ or $\pm \sqrt{-1}$.

4. Given $x^6 - 2x^3 = -1$, to find $x$. *Ans.* $x = 1$.

5. Given $x^8 - 6x^4 = 160$, to find $x$.

*Ans.* $x = \pm 2$ or $\pm 2\sqrt{-1}$.

**178.** Sometimes an equation may be solved by considering several of its terms united, as the unknown quantity.

### EXAMPLES.

1. Given $(1 + x)^2 + (1 + x) = 12$, to find $x$.

First, $1 + x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 12} = 3$ or $-4$.

Then $x = 2$ or $-5$.

2. Given $(3 + x^2)^4 - (3 + x^2)^2 = 240$, to find $x$.

First, $(3 + x^2)^2 = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 240} = 16$ or $-15$.

Then $3 + x^2 = \pm 4$ or $\pm \sqrt{-15}$.

Whence $x^2 = 1$ or $-7$, or $-3 \pm \sqrt{-15}$.

$\therefore x = \pm 1$, or $\pm \sqrt{-7}$, or $\pm \sqrt{-3 \pm \sqrt{-15}}$.

3. Given $(1 + x + x^2)^2 - 2(1 + x + x^2) = 143$, to find $x$.

First, $1 + x + x^2 = 1 \pm \sqrt{1 + 143} = 13$ or $-11$.

Then $x^2 + x = 12$ or $-12$.

$\therefore\ x = 3$, or $-4$, or $x = \frac{1}{2}(-1 \pm \sqrt{-47})$.

4. Given $(x^2 - 4x)^2 + 3(x^2 - 4x) = 0$, to find $x$.

*Ans.* $x = 4, 3$, or $1$.

5. Given $(1 + 2x + x^2)^2 - \frac{1}{8}(1 + 2x + x^2) = 254$, to find $x$.

*Ans.* $x = 3$, or $-5$, or $-1 \pm \sqrt{-15\frac{7}{8}}$.

6. Given $(x^4 + 8x^2 + 16)^4 - 2(x^4 + 8x^2 + 16)^2 = 389375$, to find $x$.

*Ans.* $x = \pm 1$, or $\pm 3\sqrt{-1}$, or $\pm\sqrt{-4 \pm \sqrt{-5}}$,

or $\pm\sqrt{-4 \pm \sqrt[4]{-623}}$.

7. Given $x^4 + x^3 + x^2 + x + 1 = 0$, to find $x$.

*Solution.*

$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0$......given equation divided by $x^2$.

$x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 = 0$......last equation rearranged.

$x^2 + 2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 = 2$......2, added to each side of the last equation.

$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 1$......last equation factored.

Whence $x + \frac{1}{x} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$

$\therefore\ x = -\frac{1}{4}(1 \mp \sqrt{5} \mp \sqrt{-10 \mp 2\sqrt{5}})$.

8. Given $x^4 - x^3 + x^2 - x + 1 = 0$, to find $x$.

*Ans.* $x = \frac{1}{4}(1 \pm \sqrt{5} \pm \sqrt{-10 \pm 2\sqrt{5}})$.

**179.** LITERAL EQUATIONS.

1. Given $x - 2ax = 2ab + b^2$, to find $x$.

*Ans.* $x = a \pm \sqrt{a^2 + 2ab + b^2} = a \pm (a + b) = 2a + b$ or $-b$.

2. Given $x^2 - 2ax = -a^2 + b^2$, to find $x$.

*Ans.* $x = a \pm \sqrt{a^2 - a^2 + b^2} = a \pm b$.

3. Given $x^2 - (a + b)x = -ab$, to find $x$.

*Ans.* $x = +\frac{a+b}{2} \pm \sqrt{\frac{a^2 + 2ab + b^2}{4} - ab} = +\frac{a+b}{2} \pm \frac{a-b}{2} = a$ or $b$.

4. Given $x^2 - (a - 1)x = a$, to find $x$. *Ans.* $x = a$, or $-1$.

5. Given $\frac{x}{x+a} = \frac{b}{x-b}$, to find $x$. *Ans.* $x = b \pm \sqrt{ab + b^2}$.

6. Given $\frac{x+a}{x-a} - b = \frac{a-x}{a+x}$, to find $x$.

*Ans.* $x = \pm \frac{a\sqrt{b+2}}{\sqrt{b-2}}$.

7. Given $\frac{a-x}{a+x} + b = \frac{a+x}{a-x}$, to find $x$.

*Ans.* $x = \frac{a}{b}(-2 \pm \sqrt{4 + b^2})$.

8. Given $a^2 + b^2 - 2bx + x^2 = \frac{m^2x^2}{n^2}$ (1), to find $x$.

$a^2n^2 + b^2n^2 - 2bn^2x + n^2x^2 = m^2x^2$, (2) = (1) cleared of fractions.

$n^2x^2 - m^2x^2 - 2bn^2x = -a^2n^2 - b^2n^2$ (3) = (2) transposed.

$$x^2 - \frac{2bn^2x}{n^2 - m^2} = \frac{-a^2n^2 - b^2n^2}{n^2 - m^2} \quad (4) = (3) \div (n^2 - m^2).$$

Then
$$x = \frac{bn^2}{n^2 - m^2} \pm \sqrt{\frac{b^2n^4}{(n^2 - m^2)^2} + \frac{-a^2n^2 - b^2n^2}{n^2 - m^2}}$$

$$x = \frac{bn^2}{n^2 - m^2} \pm \sqrt{\frac{b^2n^4 - a^2n^4 - b^2n^4 + a^2n^2m^2 + b^2m^2n^2}{(n^2 - m^2)^2}}.$$

$$x = \frac{bn^2 \pm n\sqrt{a^2m^2 + b^2m^2 - a^2n^2}}{n^2 - m^2}$$

And finally, $x = \frac{n}{n^2 - m^2}(bn \pm \sqrt{a^2m^2 + b^2m^2 - a^2n^2})$.

(*Vide* § **28**, ex. 16.)

## EQUATIONS CONTAINING RADICAL QUANTITIES.

**180.** Equations containing radical quantities are usually solved by a judicious application of Axiom VII. No rule can be invariably followed in such equations. Some general directions will be better understood after the solution of a few

### EXAMPLES.

1. Given $7\sqrt{x} + 5 = 10 + 4\sqrt{x}$ (1), to find $x$.

$3\sqrt{x} = 5.$ (2) = (1) transposed and united.

$\sqrt{x} = \frac{5}{3}.$ (3) = (2) ÷ 3.

$x = \frac{25}{9}.$ (4) = (3)². (*Vide* Ax. VII.)

2. Given $\sqrt{x-8} = \sqrt{x} - \sqrt{2}$ (1), to find $x$.

$x - 8 = x - 2\sqrt{2x} + 2.$ (2) = (1)² Ax. VII

$\sqrt{2x} = 5.$ (3) = (2) reduced.

$2x = 25.$ (4) = (3)².

Whence $x = 12\frac{1}{2}.$

3. Given $\frac{4}{x} + \frac{\sqrt{16 - x^2}}{x} = \frac{x}{4}$ (1), to find $x$.

$$\frac{\sqrt{16 - x^2}}{x} = \frac{x}{4} - \frac{4}{x} \qquad (2) = (1) \text{ transposed.}$$

$$\frac{16 - x^2}{x^2} = \frac{x^2}{16} - 2 + \frac{16}{x^2}. \qquad (3) = (2)^2$$

Whence $x = \pm 4.$ (4) = (3) reduced.

4. Given $\frac{\sqrt{x+9}}{\sqrt{x}} + \frac{6}{\sqrt{x+9}} = \frac{4\sqrt{x}}{\sqrt{x+9}}$ (1), to find $x$.

$x + 9 + 6\sqrt{x} = 4x$ (2) = (1) cleared of fractions.

$6\sqrt{x} = 3x - 9.$ (3) = (2) transposed.

$36x = 9x^2 - 54x + 81.$ (4) = (3) squared.

Whence $x = 9$ or $1.$ (5) = (4) reduced.

5. Given $\frac{x-9}{\sqrt{x}+3} + \frac{x-4}{\sqrt{x}-2} = \frac{4(x-16)}{\sqrt{x}+4}$ (1), to find $x$.

(*Vide* § **163**, ex. 32.)

$\sqrt{x} - 3 + \sqrt{x} + 2 = 4\sqrt{x} - 16$. (2) = (1) with each term reduced.

$2\sqrt{x} = 15$. (3) = (2) with terms united.

Whence $x = 56\frac{1}{4}$ (4) = $(3)^2$ and reduced.

6. Given $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{a^2 + x^2}}$ (1), to find $x$.

$x\sqrt{a^2 + x^2} + a^2 + x^2 = 2a^2$. (2) = (1) cleared of fractions.

$x\sqrt{a^2 + x^2} = a^2 - x^2$. (3) = (2) transposed.

$a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4$ (4) = $(3)^2$.

$3a^2x^2 = a^4$ (5) = (4) reduced.

Whence $x = \pm \frac{a}{3}\sqrt{3}$. (6)

7. Given $\frac{\sqrt{x + a}}{\sqrt{x}} + \frac{2\sqrt{a}}{\sqrt{x + a}} = \frac{b^2\sqrt{x}}{\sqrt{x + a}}$ (1), to find $x$.

$\frac{x + a}{x} + 2\sqrt{\frac{a}{x}} = b^2$ (2) = (1) $\times \frac{\sqrt{x + a}}{\sqrt{x}}$

$1 + \frac{a}{x} + 2\sqrt{\frac{a}{x}} = b^2$ (3) = (2) modified.

$\frac{a}{x} + 2\sqrt{\frac{a}{x}} + 1 = b^2$ (4) = (3) modified.

$\frac{\sqrt{a}}{\sqrt{x}} + 1 = \pm b$ (5) = $\sqrt{(4)}$.

$\sqrt{x} = \frac{\sqrt{a}}{b \mp 1}$. (6) = (5) reduced.

Whence $x = \frac{a}{(b \mp 1)^2}$.

8. Given $\frac{\sqrt{x} + \sqrt{x - a}}{\sqrt{x} - \sqrt{x - a}} = \frac{n^2a}{x - a}$ (1), to find $x$.

$\frac{(\sqrt{x} + \sqrt{x - a})^2}{a} = \frac{n^2a}{x - a}$. (2) = (1) modified (*Vide* § **164**, ex. 9).

$\sqrt{x} + \sqrt{x - a} = \frac{\pm na}{\sqrt{x - a}}$. (3) = (2) modified, and root taken.

$\sqrt{x^2 - ax} + x - a = \pm\, na$ (4) = (3) cleared of fractions.

$\sqrt{x^2 - ax} = a\,(1 \pm n) - x$ (5) = (4) transposed.

$x^2 - ax = a^2(1 \pm n)^2 - 2ax\,(1 \pm n) + x^2$ (6) = (5)$^2$.

Whence $x = \dfrac{a\,(1 \pm n)^2}{1 \pm 2n}$.

From the examples now given, it will have been seen that the object has been, in every instance, to relieve $x$ from its radical sign, after which its value is obtained in the usual way.

To effect the object, the terms of the equation must be so arranged that, on squaring, as many of the radicals as possible will disappear.

If, on squaring, radical terms still remain, re-arrange, and square the equation a second time.

Examples 7 and 8, above, exhibit anomalous methods of solution. They should be carefully studied,—that is, studied until the reason for each change is clearly perceived.

The pupil will find in the following examples ample opportunity to improve his powers of analysis; and we take this occasion to remind both *teacher* and *pupil*, that a day occupied in the investigation of a single equation is discreditable to no one desirous of obtaining a familiar acquaintance with the various operations of algebra. *Indeed, such examinations are absolutely necessary.*

Complicated equations can generally be solved in a variety of ways, but the *best method* can be learned only from *practice.*

As a further illustration, we will resume example 4 above, and then leave the pupil to exercise his own ingenuity.

9. Given $\dfrac{\sqrt{x+9}}{\sqrt{x}} + \dfrac{6}{\sqrt{x+9}} = \dfrac{4\sqrt{x}}{\sqrt{x+9}}$ (1), to find $x$.

$\dfrac{x+9}{x} + \dfrac{6}{\sqrt{x}} = 4.$ (2) = (1) $\times \dfrac{\sqrt{x+9}}{\sqrt{x}}$.

$$\frac{9}{x} + \frac{6}{\sqrt{x}} + 1 = 4. \qquad (3) = (2) \text{ modified.}$$

$$\frac{3}{\sqrt{x}} + 1 = \pm 2. \qquad (4) = \sqrt{(3)}.$$

Whence $x = 9$ or $1$.

From (4) we have $\sqrt{x} = 3$, or $\sqrt{x} = -1$; and it is with this limitation that the value 1 satisfies the original equation.

10. Given $17 + 2\sqrt{x^2 + 9} = 27$, to find $x$. *Ans.* $x = \pm 4$.

11. Given $5 - \sqrt{25 - x^2} = 3x$, to find $x$. *Ans.* $x = 3$.

12. Given $\sqrt{x - 32} = 16 - \sqrt{x}$, to find $x$. *Ans.* $x = 81$.

13. Given $\sqrt{x + 40} = 10 - \sqrt{x}$, to find $x$. *Ans.* $x = 9$.

14. $\sqrt{x - 16} = \sqrt{x} - 2$. *Ans.* 25.

15. $\sqrt{x + 8} - \sqrt{x - 8} = 2\sqrt{2}$. *Ans.* 10.

16. $\sqrt{x} + \sqrt{x - 9} = \dfrac{9}{\sqrt{x - 9}}$. *Ans.* 12.

*17. $\sqrt{1 + x\sqrt{x^2 - 1}} = 1 - x$. *Ans.* $\frac{5}{4}$.

18. $\sqrt{x} - \sqrt{10 - x} = \dfrac{\sqrt{x} + \sqrt{10 - x}}{2}$. *Ans.* 9.

19. $\dfrac{9x - 1}{\sqrt{9x} + 1} = 4 + \dfrac{\sqrt{9x} - 1}{2}$. *Ans.* 9.

20. $\dfrac{3\sqrt{2x} + 10}{3\sqrt{2x} - 10} = \dfrac{\sqrt{2x} + 16}{\sqrt{2x} - 4}$. *Ans.* $4\frac{1}{2}$.

21. $\dfrac{\sqrt{x} + 28}{\sqrt{x} + 4} = \dfrac{\sqrt{x} + 38}{\sqrt{x} + 6}$. *Ans.* 4.

22. $\dfrac{a}{x} + \dfrac{\sqrt{a^2 - x^2}}{x} = \dfrac{x}{b}$. *Ans.* $\pm\sqrt{2ab - b^2}$.

*23. $\dfrac{2\sqrt{x} + a}{\sqrt{x} + 2a} = \dfrac{2a - \sqrt{x}}{\sqrt{x}}$. *Ans.* $a^2$ or $\dfrac{16a^2}{9}$.

24. $\dfrac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} = b$. *Ans.* $\pm\dfrac{2a\sqrt{b}}{1 + b}$.

25. $\dfrac{\sqrt{a+x}}{\sqrt{x}}+\dfrac{\sqrt{a-x}}{\sqrt{x}}=\dfrac{\sqrt{x}}{\sqrt{b}}$. *Ans.* $\pm 2\sqrt{ab-b^2}$.

26. $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x}=b$. *Ans.* $\dfrac{\pm a(1\pm\sqrt{2b-b^2})}{\sqrt{2b-b^2}}$.

*27. $\dfrac{5-\sqrt{25-x^2}}{5+\sqrt{25-x^2}}=\frac{1}{4}$. *Ans.* $\pm 4$.

*28. $\dfrac{\sqrt{8+x}}{\sqrt{x}}+\dfrac{\sqrt{8-x}}{\sqrt{x}}=\dfrac{\sqrt{x}}{2}$. *Ans.* $\pm 8$.

*29. $\dfrac{\sqrt{20}-x}{\sqrt{20}+x}+\sqrt{5}=\dfrac{\sqrt{20}+x}{\sqrt{20}-x}$. *Ans.* 2 or $-10$.

30. $\dfrac{\sqrt{4x+20}}{4+\sqrt{x}}=\dfrac{4-\sqrt{x}}{\sqrt{x}}$. *Ans.* 4 or $-\dfrac{64}{3}$.

31. $\dfrac{\sqrt{x+1}}{\sqrt{x-1}}+\dfrac{\sqrt{x-1}}{\sqrt{x+1}}=a$. *Ans.* $\pm\dfrac{a}{\sqrt{a^2-4}}$.

32. $\sqrt{a^2+ax}=a-\sqrt{a^2-ax}$. *Ans.* $\pm\frac{a}{2}\sqrt{3}$.

33. $\sqrt{a}+\sqrt{x}=\sqrt{ax}$. *Ans.* $\dfrac{a}{(\sqrt{a}-1)^2}$.

## PROBLEMS

### INVOLVING EQUATIONS OF THE SECOND DEGREE.

**181.** 1. Three times the square of a number added to four times the number is equal to 64. What is the number?

$$3x^2+4x=64. \qquad \textit{Ans. } 4 \text{ or } -5\tfrac{1}{3}.$$

2. A man bought a number of sheep for \$200, and, reserving 20, he sold the remainder for \$150, gaining \$1 on the price of each sheep. What number was purchased?

Let $x=$ the number.

Then $\dfrac{200}{x}=$ price per head of those bought.

and $\dfrac{150}{x-20}=$ price per head of those sold.

Now, by the question the latter price is \$1 more than the former. Hence,

$$\frac{200}{x} + 1 = \frac{150}{x - 20}. \qquad \textit{Ans. } 50 \text{ sheep.}$$

3. Divide 50 into two parts so that their product may be 621

*Ans.* 23 and 27.

4. The difference of two numbers is 6, and their product is 216. What are the numbers? *Ans.* 12, 18.

5. A man sold a watch for \$75, and gained as much per cent. as the watch cost him. What did he pay for it? *Ans.* \$50.

6. A man sold a watch for \$24, and lost as much per cent. as the watch cost him. What did he pay for it? *Ans.* \$40 or \$60.

7. If 7 be added to a certain number and 3 be subtracted, the product of the sum and difference will be 119. What is the number? *Ans.* 10 or — 14.

8. A merchant bought a quantity of flour for \$72. Had he bought 6 barrels more for the same sum, the price per barrel would have been \$1 less. How many barrels did he buy, and at what price per barrel? *Ans.* 18 barrels, at \$4 per barrel.

9. If a certain number is subtracted from 12 and the remainder is multiplied by the number, the product will be 35. What is the number? *Ans.* 5 or 7.

10. If a certain number be divided by 10, and this quotient be added to the quotient of 10 divided by the number, the sum will be $3\frac{1}{3}$. What is the number? *Ans.* 30 or $3\frac{1}{3}$.

11. A man travelled 105 miles, and then found that if he had gone 2 miles less per hour he would have been 6 hours longer on his journey. At what rate did he travel per hour?

*Ans.* 7 miles.

12. Divide 40 into two parts so that the sum of their squares may be 1000. *Ans.* 30 and 10.

13. Two fields differing in quantity by 10 acres were each

sold for \$2800, one bringing \$5 per acre more than the other. What was the number of acres in each? *Ans.* 70 and 80 acres.

14. The product of two numbers is 120. If 2 be added to the less and 3 be subtracted from the greater, the product of the sum and difference will still be 120. What are the numbers? *Ans.* 8 and 15.

15. Two men are travelling towards each other. On meeting, B has travelled 20 miles farther than A. A, by preserving his rate of travel, will go the distance B has already travelled in 20 hours; but B will be only 15 in passing over A's distance (at his former rate). What is the rate per hour of each?

*Ans.* A, 7.464, and B, 8.61$\frac{13}{15}$.

16. Two merchants sold the same kind of stuff, and together received \$35. The second sold 3 yards more than the first. Had the prices per yard been interchanged, the first would have received \$24 and the second \$12$\frac{1}{2}$, gaining thereby \$1$\frac{1}{2}$. How many yards were sold by each, and at what price per yard?

*Ans.* 15 at \$1$\frac{1}{3}$ and 18 at \$$\frac{5}{6}$, or 5 at \$3 and 8 at \$2$\frac{1}{2}$.

17. Divide the number 10 into two parts so that 10 times the second part may be the square of the first part.

*Ans.* $5(-1+\sqrt{5})$ and $5(3-\sqrt{5}.)$

18. Divide the number $a$ into two parts so that the square of the second part may be the first multiplied by $a$.

*Ans.* $\frac{a}{2}(3 \pm \sqrt{5})$ and $\frac{a}{2}(-1 \mp \sqrt{5})$.

19. A and B travel at the same rate towards Washington. At the 50th mile-stone from Washington, A *overtakes* a flock of geese travelling 1$\frac{1}{2}$ miles an hour, and two hours afterwards meets a coach travelling 2$\frac{1}{4}$ miles per hour; B *overtakes* the geese at the 45th mile-stone, and *meets* the coach 40 minutes before reaching the 31st mile-stone. What is the distance between A and B? *Ans.* 25 miles.

## EQUATIONS WITH TWO UNKNOWN QUANTITIES.

**182.** 1. Given $\left.\begin{array}{ll} x + y = 8 & (1), \\ \text{and} \quad xy = 15 & (2), \end{array}\right\}$ to find $x$ and $y$.

$$
\begin{aligned}
x^2 + 2xy + y^2 &= 64, \ (3) = (1)^2. \\
4xy \quad &= 60, \ (4) = (2) \times 4. \\
x^2 - 2xy + y^2 &= 4, \quad (5) = (3) - (4). \\
x - \quad y \quad &= 2, \quad (6) = \sqrt{(5)}. \\
x \quad &= 5, \quad (7) = ((1) + (6)) \div 2. \\
y \quad &= 3, \ ((8) = (1) - (6)) \div 2.
\end{aligned}
$$

The above operation will be readily understood, and the *object* of each step. In the same way solve and verify the following:—

2. $\begin{pmatrix} x+y=10. \\ xy=16. \end{pmatrix}$ 3. $\begin{pmatrix} x+y=12. \\ xy=32. \end{pmatrix}$ 4. $\begin{pmatrix} x+y=20. \\ xy=64. \end{pmatrix}$ 5. $\begin{pmatrix} x+y=50. \\ xy=400. \end{pmatrix}$

(6.) $\begin{pmatrix} x+y=3\frac{1}{2}. \\ xy=1\frac{1}{2}. \end{pmatrix}$ (7.) $\begin{pmatrix} x+y=-1. \\ xy=-56. \end{pmatrix}$ (8.) $\begin{pmatrix} x+y=\ 14. \\ xy=+45. \end{pmatrix}$ (9.) $\begin{pmatrix} x+y=\ 2. \\ xy=-63. \end{pmatrix}$

**183.** (1.) Given $\left.\begin{array}{ll} x + y = a & (1), \\ \text{and} \quad xy = b & (2), \end{array}\right\}$ to find $x - y$.

$$
\begin{aligned}
x^2 + 2xy + y^2 &= a^2, (3) = (1)^2. \\
4xy \quad &= 4b, (4) = (2) \times 4. \\
x^2 - 2xy + y^2 &= a^2 - 4b, \ (5) = (3) - (4). \\
x - y &= \sqrt{a^2 - 4b}, (6) = \sqrt{(5)}.
\end{aligned}
$$

Hence, when the sum and product of two numbers are given, take the square root of four times the product *subtracted* from the square of the sum, and this root will be the difference of the numbers.

(2.) Given $x + y = 10$, and $xy = 24$, to find $x$ and $y$.

By the rule $x - y = \sqrt{10^2 - 4 \times 24} = 2$.

Hence, $\quad x = 6$ and $y = 4$.

| (3.) | (4.) | (5.) | (6.) |
|---|---|---|---|
| $(x + y = 13.)$ | $(x + y = 11.)$ | $(x + y = 20.)$ | $(x + y = 5\frac{1}{2}.)$ |
| $(\quad xy = 24.)$ | $(\quad xy = 28.)$ | $(\quad xy = 96.)$ | $(\quad xy = 2\frac{1}{2}.)$ |

**184.** 1. Given $\left.\begin{array}{ll} x - y = 8 & (1), \\ \text{and} \quad xy = 48 & (2), \end{array}\right\}$ to find $x$ and $y$.

$$x^2 - 2xy + y^2 = 64, \quad (3) = (1)^2.$$
$$4xy = 192, \ (4) = (2) \times 4.$$
$$x^2 + 2xy + y^2 = 256, \ (5) = (4) + (3).$$
$$x + y = 16, \quad (6) = \sqrt{(5)}.$$

Hence $x = 12$ and $y = 4$.

In the same manner solve the following equations.

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $(x - y = 4.\ )$ | $(x - y = 3.\ )$ | $(x - y = 5.\ )$ | $(x - y = 3\frac{1}{2}.)$ |
| $(\quad xy = 21.)$ | $(\quad xy = 70.)$ | $(\quad xy = 300.)$ | $(\quad xy = 2.\ )$ |
| **(6.)** | **(7.)** | **(8.)** | **(9.)** |
| $(x - y = -2.)$ | $(x - y = 7\frac{2}{3}.)$ | $(x - y = 1.\ )$ | $(x - y = 11.)$ |
| $(\quad xy = \ 24.)$ | $(\quad xy = 2\frac{2}{3}.)$ | $(\quad xy = 3\frac{1}{9}.)$ | $(\quad xy = 26.)$ |

10. Given $\left.\begin{array}{ll} x - y = a & (1), \\ \text{and} \quad xy = b & (2), \end{array}\right\}$ to find $x + y$.

$$x^2 - 2xy + y^2 = a^2, \ (3) = (1)^2.$$
$$4xy = 4b, \ (4) = (2) \times 4.$$
$$x^2 + 2xy + y^2 = a^2 + 4b, \ (5) = (4) \pm (3).$$
$$x + y = \sqrt{a^2 + 4b}, \ (6) = \sqrt{(5)}.$$

Hence, when the difference and product of two numbers are given, take the square root of four times the product *added* to the square of the difference, and this root will be the sum of the numbers.

11. Given $x - y = 3$, and $xy = 28$, to find $x$ and $y$.

By the *rule*, $\quad x + y = \sqrt{9 + 28 \times 4} = 11.$

Hence $\quad x = 7$ and $y = 4$.

| (12.) | (13.) | (14.) | (15.) |
|---|---|---|---|
| $\begin{cases} x - y = 10. \\ xy = 119. \end{cases}$ | $\begin{cases} x - y = 5. \\ xy = 24. \end{cases}$ | $\begin{cases} x - y = 2\frac{1}{2}. \\ xy = 1\frac{1}{2}. \end{cases}$ | $\begin{cases} x - y = 5\frac{1}{2}. \\ xy = 3. \end{cases}$ |

**185.** 1. Given $\left.\begin{aligned} x^2 + y^2 &= 25 \ (1), \\ \text{and} \quad x + y &= 7 \ \ (2), \end{aligned}\right\}$ to find $x$ and $y$.

$$x^2 + 2xy + y^2 = 49, \quad (3) = (2)^2.$$
$$2xy = 24, \quad (4) = (3) - (1).$$
$$x^2 - 2xy + y^2 = 1, \quad (5) = (1) - (4).$$
$$x - y = 1, \quad (6) = \sqrt{(5)}.$$

Hence $x = 4$ and $y = 3$.

In the same way solve the following equations.

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $x^2 + y^2 = 50.$ | $x^2 + y^2 = 5.$ | $x^2 + y^2 = 29.$ | $x^2 + y^2 = 40.$ |
| $x + y = 8.$ | $x + y = 3.$ | $x + y = 7.$ | $x + y = 8.$ |

6. Given $\left.\begin{aligned} x + y &= a \ (1), \\ \text{and} \quad x^2 + y^2 &= c \ (2), \end{aligned}\right\}$ to find $x$ and $y$.

$$x^2 + 2xy + y^2 = a^2, \quad (3) = (1)^2.$$
$$2xy = a^2 - c, \ (4) = (3) - (2).$$
$$x^2 - 2xy + y^2 = 2c - a^2, (5) = (2) - (4).$$
$$x - y = \sqrt{2c - a^2}, (6) = \sqrt{(5)}.$$

Hence, when the sum of two numbers is given, and also the sum of their squares, take the square root of the square of the sum subtracted from twice the sum of their squares, and this root will be the difference of the numbers.

7. Given $x^2 + y^2 = 89$, and $x + y = 13$, to find $x$ and $y$.

By the *rule*, $x - y = \sqrt{89 \times 2 - 13^2} = 3.$

Hence $x = 8$ and $y = 5$.

| (8.) | (9) | (10.) | (11.) |
|---|---|---|---|
| $\begin{cases} x^2+y^2=50\frac{1}{2}. \\ x+y=10. \end{cases}$ | $\begin{cases} x^2+y^2=25. \\ x+y=7. \end{cases}$ | $\begin{cases} x^2+y^2=58. \\ x+y=-10. \end{cases}$ | $\begin{cases} x^2+y^2=41. \\ x+y=-1. \end{cases}$ |

| (12.) | (13.) | (14.) | (15.) |
|---|---|---|---|
| $(x^2+y^2=1105.)$ | $(x^2+y^2=34.)$ | $(x^2+y^2=65.)$ | $(x^2+y^2=10.)$ |
| $(x+y=47.)$ | $(x+y=2.)$ | $(x+y=9.)$ | $(x+y=3.)$ |

**186.** 1. Given $\left.\begin{array}{ll} x^2+y^2=52 & (1), \\ \text{and} \quad x-y=2 & (2), \end{array}\right\}$ to find $x$ and $y$.

$$x^2+2xy+y^2=4, \qquad (3)=(2)^2.$$
$$2xy=48, \qquad (4)=(1)-(3).$$
$$x^2+2xy+y^2=100, \qquad (5)=(1)+(4).$$
$$+y=\pm 10, \quad (6)=\sqrt{(5)}.$$

Hence $x=6$ or $-4$, and $y=4$ or $-6$.

In the same way solve the following equations.

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $(x^2+y^2=25.)$ | $(x^2+y^2=41.)$ | $(x^2+y^2=65.)$ | $(x^2+y^2=61.)$ |
| $(x-y=1.)$ | $(x-y=1.)$ | $(x-y=3.)$ | $(x-y=1.)$ |

6. Given $\left.\begin{array}{ll} x-y=d & (1), \\ \text{and} \quad x^2+y^2=c & (2), \end{array}\right\}$ to find $x$ and $y$.

$$x^2-2xy+y^2=d^2$$
$$2xy=c-d^2$$
$$x^2+2xy+y^2=2c-d^2$$
$$x+y=\pm\sqrt{2c-d^2}.$$

Hence, when the difference of two numbers is given, and also the sum of their squares, take the square root of the square of the difference subtracted from twice the sum of their squares, and this root will be the sum of the numbers.

| (7.) | (8.) | (9.) | (10.) |
|---|---|---|---|
| $(x^2+y^2=74.)$ | $(x^2+y^2=45.)$ | $(x^2+y^2=65.)$ | $(x^2+y^2=65.)$ |
| $(x-y=2.)$ | $(x-y=3.)$ | $(x-y=3.)$ | $(x-y=11.)$ |

**187.** 1. Given $\left.\begin{array}{ll} x^2-y^2=17 & (1), \\ \text{and} \quad x-y=2 & (2), \end{array}\right\}$ to find $x$ and $y$.

$$x+y=8\tfrac{1}{2}, (3)=(1)\div(2.)$$

Hence $x=5\frac{1}{4}$, and $y=3\frac{1}{4}$.

In the same way solve the equations—

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $\begin{cases} x^2 - y^2 = 55. \\ x - y = 11. \end{cases}$ | $\begin{cases} x^2 - y^2 = 12. \\ x - y = 3. \end{cases}$ | $\begin{cases} x^2 - y^2 = 18. \\ x - y = 1\frac{1}{2}. \end{cases}$ | $\begin{cases} x^2 - y^2 = 14. \\ x - y = 5. \end{cases}$ |
| **(6.)** | **(7.)** | **(8.)** | **(9.)** |
| $\begin{cases} x^2 - y^2 = 15. \\ x - y = 10. \end{cases}$ | $\begin{cases} x^2 - y^2 = 26. \\ x - y = 13. \end{cases}$ | $\begin{cases} x^2 - y^2 = 15. \\ x - y = 3. \end{cases}$ | $\begin{cases} x^2 - y^2 = 30. \\ x - y = 60. \end{cases}$ |

**188.** 1. Given $x^2 - y^2 = 15 \ (1),$ and $x + y = 15 \ (2),$ to find $x$ and $y$.

$$x - y = 1, \quad (3) = (1) \div (2).$$

Hence $\quad x = 8$, and $y = 7$.

In the same way solve the equations—

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $\begin{cases} x^2 - y^2 = 18. \\ x + y = 9. \end{cases}$ | $\begin{cases} x^2 - y^2 = 27. \\ x + y = -13\frac{1}{2}. \end{cases}$ | $\begin{cases} x^2 - y^2 = 53. \\ x + y = 17\frac{2}{3}. \end{cases}$ | $\begin{cases} x^2 - y^2 = 500. \\ x - y = 125. \end{cases}$ |

6. Given $x^2 - y^2 = m,$ and $x - y = d,$ to find $x$ and $y$.

*Ans.* $x = \dfrac{m + d^2}{2d}, \ y = \dfrac{m - d^2}{2d}.$

7. Given $x^2 - y^2 = m,$ and $x + y = a,$ to find $x$ and $y$.

*Ans.* $x = \dfrac{a^2 + m}{2a}, \ y = \dfrac{a^2 - m}{2a}.$

**189.** 1. Given $x^2 + y^2 = 5 \ (1),$ and $xy = 2 \ (2),$ to find $x$ and $y$.

$$2xy = 4, \quad (3) = (2) \times 2.$$

$$x^2 + 2xy + y^2 = 9, \quad (4) = (1) + (3).$$

$$x^2 - 2xy + y^2 = 1, \quad (5) = (1) - (3).$$

$$x + y = \pm 3, \quad (6) = \sqrt{(4)}.$$

$$x - y = \pm 1, \quad (7) = \sqrt{(5)}.$$

Hence $\quad x = \pm 2$ and $y = \pm 1$.

In the same way solve the equations—

$$\begin{array}{cccc}
(2.) & (3.) & (4.) & (5.) \\
\left(\begin{array}{l} x^2+y^2=10. \\ \quad xy=3. \end{array}\right) &
\left(\begin{array}{l} x^2+y^2=13. \\ \quad xy=6. \end{array}\right) &
\left(\begin{array}{l} x^2+y^2=18\frac{1}{2}. \\ \quad xy=8\frac{3}{4}. \end{array}\right) &
\left(\begin{array}{l} x^2+y^2=13 \\ \quad xy=-6. \end{array}\right) \\
(6.) & (7.) & (8.) & (9.) \\
\left(\begin{array}{l} x^2+y^2=50\frac{1}{2}. \\ \quad xy=-24\frac{3}{4}. \end{array}\right) &
\left(\begin{array}{l} x^2+y^2=16\frac{5}{9}. \\ \quad xy=-7\frac{7}{9}. \end{array}\right) &
\left(\begin{array}{l} x^2+y^2=50. \\ \quad xy=25. \end{array}\right) &
\left(\begin{array}{l} x^2+y^2=25. \\ \quad xy=12. \end{array}\right)
\end{array}$$

10. Given $x^2+y^2=c$, and $xy=b$, to find $x$ and $y$.

$$Ans.\ x=\frac{\pm\sqrt{c+2b}\pm\sqrt{c-2b}}{2},\ y=\frac{\pm\sqrt{c+2b}\pm\sqrt{c-2b}}{2}.$$

**190.** 1. Given $\left.\begin{array}{l} x^2-y^2=7 \quad (1), \\ \qquad xy=12 \ (2), \end{array}\right\}$ to find $x$ and $y$.

$$\begin{array}{rcll}
x^4-2x^2y^2+y^4 &=& 49, & (3)=(1)^2. \\
4x^2y^2 &=& 576, & (4)=4\times(2)^2. \\
x^4+2x^2y^2+y^4 &=& 625, & (5)=(3)+(4). \\
x^2+y^2 &=& \pm 25, & (6)=\sqrt{(5)}.
\end{array}$$

Hence $x=\pm 4$, or $\pm 3\sqrt{-1}$, and $y=\pm 3$, or $\pm 4\sqrt{-1}$.

In the same way solve and verify the following equations.

$$\begin{array}{cccc}
(2.) & (3.) & (4.) & (5.) \\
\left(\begin{array}{l} x^2-y^2=24. \\ \quad xy=5. \end{array}\right) &
\left(\begin{array}{l} x^2-y^2=21. \\ \quad xy=10. \end{array}\right) &
\left(\begin{array}{l} x^2-y^2=16. \\ \quad xy=15. \end{array}\right) &
\left(\begin{array}{l} x^2-y^2=40. \\ \quad xy=21. \end{array}\right) \\
(6.) & (7.) & (8.) & (9.) \\
\left(\begin{array}{l} x^2-y^2=60. \\ \quad xy=16. \end{array}\right) &
\left(\begin{array}{l} x^2-y^2=80. \\ \quad xy=9. \end{array}\right) &
\left(\begin{array}{l} x^2-y^2=1. \\ \quad xy=\sqrt{6}. \end{array}\right) &
\left(\begin{array}{l} x^2-y^2=3. \\ \quad xy=\sqrt{10}. \end{array}\right)
\end{array}$$

## *REVIEW.*

**191.**

(1.)

$$x^2-y^2=19.$$
$$x+y=19.$$

*Ans.* $x=10,\ y=9.$

(2.)

$$x^2-y^2=16.$$
$$x-y=2.$$

*Ans.* $x=5,\ y=3.$

(3.)

$$x^2 - y^2 = 15.$$
$$x - y = 3.$$

*Ans.* $x = 4, y = 1.$

(4.)

$$x^2 - y^2 = 120.$$
$$x - y = 10.$$

*Ans.* $x = 11, y = 1.$

(5.)

$$x^2 - y^2 = 39.$$
$$x + y = 13.$$

*Ans.* $x = 8, y = 5.$

(6.)

$$x^2 - y^2 = 6.$$
$$x + y = 6.$$

*Ans.* $x = 3\frac{1}{2}, y = 2\frac{1}{2}.$

(7.)

$$x^2 - y^2 = 40.$$
$$x + y = 10.$$

*Ans.* $x = 7, y = 3.$

(8.)

$$x^2 - y^2 = 45.$$
$$x + y = 9.$$

*Ans.* $x = 7, y = 2.$

(9.)

$$x + y = 8.$$
$$xy = 15.$$

*Ans.* $x = 5, y = 3.$

(10.)

$$x - y = 1.$$
$$xy = 6.$$

*Ans.* $x = 3, y = 2.$

(11.)

$$x^2 + y^2 = 169.$$
$$x + y = 17.$$

*Ans.* $x = 12, y = 5.$

(12.)

$$x^2 - y^2 = 16.$$
$$x + y = 8.$$

*Ans.* $x = 5, y = 3.$

(13.)

$$x^2 - y^2 = 21.$$
$$x - y = 1.$$

*Ans.* $x = 11, y = 10.$

(14.)

$$x^2 + y^2 = 325.$$
$$xy = 150.$$

*Ans.* $x = 15, y = 10.$

(15.)

$$x^2 - y^2 = 33.$$
$$xy = 272.$$

*Ans.* $x = \pm 17, \text{ or } \pm 16\sqrt{-1},$
$y = \pm 16, \text{ or } \pm 17\sqrt{-1}.$

(16.)

$$x^2 + y^2 = 1300.$$
$$x - y = 10.$$

*Ans.* $x = 30, y = 20.$

(17.)

$$x^2 + y^2 = 1\tfrac{1}{144}.$$
$$xy = -\tfrac{1}{2}.$$

*Ans.* $x = \frac{3}{4}, y = -\frac{2}{3}.$

(18.)

$$x^2 - y^2 = -9.$$
$$x - y = -1.$$

*Ans.* $x = 4, y = 5.$

(19.)

$x^2 + y^2 = 10\frac{1}{2}.$

$x - y = 1.$

*Ans.* $x = \frac{1}{2} + \sqrt{5},$

$y = \sqrt{5} - \frac{1}{2}.$

(20.)

$x^2 - y^2 = 1.$

$xy = 2\sqrt{3}.$

*Ans.* $x = \pm 2$, or $\pm \sqrt{-3}.$

$y = \pm \sqrt{3}$, or $\pm 2\sqrt{-1}.$

(21.)

$x^2 + y^2 = 44.$

$xy = 3\sqrt{51}.$

*Ans.* $x = \pm 3\sqrt{3}$, or $\pm \sqrt{17}.$

$y = \pm \sqrt{17}$, or $\pm 3\sqrt{3}.$

(22.)

$x^2 + y^2 = 26.$

$x - y = \sqrt{2}.$

*Ans.* $x = 3\sqrt{2}.$

$x = 2\sqrt{2}.$

(23.)

$x^2 + y^2 = 123.$

$x - y \sqrt{3}.$

*Ans.* $x = 5\sqrt{3}.$

$y = 4\sqrt{3}.$

(24.)

$x^2 - y^2 = 45.$

$x - y = \sqrt{5}.$

*Ans.* $x = 5\sqrt{5}.$

$y = 4\sqrt{5}.$

**192.** 1. Given $\left.\begin{array}{l} x^3 + y^3 = 9 \ (1), \\ \text{and} \quad x + y = 3 \ (2), \end{array}\right\}$ to find $x$ and $y$.

FIRST METHOD.

$$\begin{array}{rll} x^3 + 3x^2y + 3xy^2 + y^3 = 27, & (3) = (2)^3. \\ 3x^2y + 3xy^2 = 18, & (4) = (3) - (1). \\ xy\,(x + y) = 6. & (5) = (4) \div 3 \text{ and factored.} \\ xy = 2. & (6) = (5) \div (2). \\ \therefore \quad x - y = 1. & \textit{Vide} \ \S\ \mathbf{183}. \end{array}$$

Hence $x = 2$ and $y = 1$.

In the same way solve and verify the following equations.

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $(x^3 + y^3 = 35.)$ | $(x^3 + y^3 = 91.)$ | $(x^3 + y^3 = 341.)$ | $(x^3 + y^3 = 65.)$ |
| $(x + y = 5.)$ | $(x + y = 7.)$ | $(x + y = 11.)$ | $(x + y = 5.)$ |

*Answers.*

$x = 3, y = 2.$ $\quad x = 4, y = 3.$ $\quad x = 6, y = 5.$ $\quad x = 4, y = 1.$

**193.** 1. Given $\left.\begin{array}{l} x^3 + y^3 = 9 \ (1), \\ x + y = 3 \ (2), \end{array}\right\}$ to find $x$ and $y$.

SECOND METHOD.

$$x^2 - xy + y^2 = 3, \quad (3) = (1) \div (2).$$
$$x^2 + 2xy + y^2 = 9, \quad (4) = (2)^2.$$
$$xy = 2, \quad (5) = \overline{(4) - (3)} \div 3.$$
$$\therefore \quad x - y = 1, \quad \textit{Vide } \S \ \mathbf{183}.$$
Hence $\quad x = 2$ and $y = 1$.

In the same way solve and verify the following equations.

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $(x^3+y^3=133.)$ | $(x^3+y^3=217.)$ | $(x^3+y^3=520.)$ | $(x^3+y^3=730.)$ |
| $(x+y=7.)$ | $(x+y=7.)$ | $(x+y=10.)$ | $(x+y=10.)$ |

*Answers*

| | | | |
|---|---|---|---|
| $x = 2, y = 5.$ | $x = 6, y = 1.$ | $x = 8, y = 2.$ | $x = 9, y = 1$ |

**194.** 2. Given $\left.\begin{array}{l} x^3 + y^3 = 9 \ (1), \\ x + y = 3 \ (2), \end{array}\right\}$ to find $x$ and $y$.

THIRD METHOD.

$x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)$, (3). *Vide* § **74**, ex. 7, and § **132** (2).

$$\therefore \quad (x + y)^3 - 3xy\,(x + y) = 9, \ (4).$$
That is, $\quad 27 - 9xy = 9$, (5) since $x + y = 3$.
$$xy = 2, \ (6).$$
$$\therefore \quad x - y = 1, \ (7). \quad \S \ \mathbf{183}.$$
Hence $\quad x = 2$, and $y = 1$.

In the same way solve the equations of § **192**, and § **193**, and also—

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $(x^3 - y^3 = 61.)$ | $(x^3 - y^3 = 342.)$ | $(x^3 - y^3 = 485.)$ | $(x^3 - y^3 = 7.)$ |
| $(x - y = 1.)$ | $(x - y = 6.)$ | $(x - y = 5.)$ | $(x - y = 1.)$ |

**195.** 1. Given $\left.\begin{array}{l} x^3 + y^3 = 9 \ (1), \\ x + y = 3 \ (2), \end{array}\right\}$ to find $x$ and $y$.

FOURTH METHOD.

$$x = 3 - y. \quad (3) = (2) \text{ transposed.}$$
$$x^3 = 27 - 27y + 9y^2 - y^3 \quad (4) = (3)^3.$$
$$\therefore \quad 27 - 27y + 9y^2 - y^3 + y^3 = 9. \quad (5) \text{ by substitution.}$$
Hence $x = 2$, and $y = 1$.

In the same way solve the equations—

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $(x^3 + y^3 = 35.)$ | $(x^3 + y^3 = 91.)$ | $(x^3 + y^3 = 341.)$ | $(x^3 - y^3 = 19.)$ |
| $(x + y = 5.)$ | $(x + y = 7.)$ | $(x + y = 11.)$ | $(x - y = 1.)$ |

**196.** 1. Given $\left.\begin{array}{l} x^3 + y^3 = 9 \ (1), \\ x + y = 3 \ (2), \end{array}\right\}$ to find $x$ and $y$.

FIFTH METHOD.

Let $x = a + b$, and $y = a - b$.

Then $x + y = 2a = 3$, and $a = \frac{3}{2}$. By addition and (2).
$$x^3 = (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$
$$y^3 = (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$
$$x^3 + y^3 = 2a^3 + 6ab^2 = 9. \quad \text{By addition.}$$
$$\therefore \quad \tfrac{27}{4} + 9b^2 = 9. \quad \text{Since } a = \tfrac{3}{2}.$$
Hence $b = \frac{1}{2}$.

But $x = a + b = \frac{3}{2} + \frac{1}{2} = 2$, and $y = a - b = \frac{3}{2} - \frac{1}{2} = 1$.

In the same way solve the equations in the preceding sections, and also

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $(x^3+y^3=28.)$ | $(x^3-y^3=26.)$ | $(x^3-y^3=7000.)$ | $(x^3+y^3=9000.)$ |
| $(x+y=4.)$ | $(x-y=2.)$ | $(x-y=10.)$ | $(x+y=30.)$ |

**197.** 1. Given $\left.\begin{array}{l} x^3 + y^3 = a \ (1), \\ x + y = b \ (2), \end{array}\right\}$ to find $x$ and $y$.

$$x^2 - \quad xy + y^2 = \frac{a}{b}, \quad (3) = (1) \div (2).$$

$$x^2 + 2xy + y^2 = b^2, \quad (4) = (2)^2.$$

$$xy = \frac{b^3 - a}{3b}, \quad (5) = \overline{(4) - (3)} \div 3.$$

$$\therefore \quad x - y = \pm\sqrt{\frac{4a - b^3}{3b}}.$$

Hence $\quad x = \frac{1}{2}\left(b \pm \sqrt{\frac{4a - b^3}{3b}}\right)$, and $y = \frac{1}{2}\left(b \mp \sqrt{\frac{4a - b^3}{3b}}\right)$.

*Vide* § **28**, ex. 19.

2. Given $x^3 - y^3 = a,$ and $x - y = b,$ $\Big\}$ to find $x$ and $y$.

*Ans.* $x = \frac{1}{2}\left(b \pm \sqrt{\frac{4a - b^3}{3b}}\right)$, and $y = \frac{1}{2}\left(- b \pm \sqrt{\frac{4a - b^3}{3b}}\right)$.

Apply these formulas to all the equations in § **192** and § **196**, inclusive.

**198.** 1. Given $x^4 + y^4 = 17$ (1), and $x + y = 3$ (2), $\Big\}$ to find $x$ and $y$.

$$x^4 + y^4 = (x + y)^4 - 4xy\,(x + y)^2 + 2x^2y^2 = 17. \quad (3)$$

*Vide* § **132** (2).

$\therefore \qquad 81 - 36xy + 2x^2y^2 = 17. \quad$ (4) Since $x + y = 3$.

Hence $\qquad xy = 2$, or 16. $\quad (5) = (4)$ reduced.

Then $\qquad x - y = 1$, or $\sqrt{-55}$.

$\therefore \; x = 2$ or $\frac{1}{2}(3 + \sqrt{-55})$, and $y = 1$ or $\frac{1}{2}(3 - \sqrt{-55})$.

In the same way solve the equations—

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $(x^4+y^4=82.)$ | $(x^4+y^4=626.)$ | $(x^4+y^4=1297.)$ | $(x^4+y^4=\frac{17}{256}.)$ |
| $(x+y=4.)$ | $(x+y=6.)$ | $(x+y=7.)$ | $(x+y=\frac{3}{4}.)$ |

6. Given $x^4 + y^4 = a$, and $x + y = b$, to find $x$ and $y$.

*Ans.* $\quad x = \frac{1}{2}\left(b \pm \sqrt{-3b^2 \mp 2\sqrt{2a + 2b^4}}\right).$

$\qquad\quad y = \frac{1}{2}\left(b \mp \sqrt{-3b^2 \mp 2\sqrt{2a + 2b^4}}\right).$

Apply these formulas to the above examples. (*Vide* **28**, ex. 24.)

**199.** 1. Given $\left.\begin{array}{ll} x^5 - y^5 = 992 & (1), \\ \text{and} \quad x - y = 2 & (2), \end{array}\right\}$ to find $x$ and $y$.

$x^5 - y^5 = (x-y)^5 + 5xy\,(x-y)^3 + 5x^2y^2\,(x-y) = 992.$
(3) *Vide* § **132** (2).

$\therefore \quad 32 + 40xy + 10x^2y^2 = 992.$ (4) Since $x - y = 2.$

Then $\quad xy = 8$, or $-12$.

Hence $x = 4$ or $-3$, or $1 \pm \sqrt{-11}$, and $y = 3$ or $-4$, or $-1 \pm \sqrt{-11}$. *Vide* § **167**, ex. 17.

In the same way solve the equations—

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $(x^5+y^5=33.)$ | $(x^5-y^5=7\frac{9}{16}.)$ | $(x^5+y^5=1056.)$ | $(x^5-y^5=781.)$ |
| $(x+y=3.)$ | $(x-y=1.)$ | $(x+y=6.)$ | $(x-y=1.)$ |

6. Given $x^5 + y^5 = a$, and $x + y = b$, to find $x$ and $y$.

$$x = \tfrac{1}{2}\left(b \pm \sqrt{-b^2 \mp 2\sqrt{\frac{4a+b^5}{5b}}}\right)$$

$$y = \tfrac{1}{2}\left(b \mp \sqrt{-b^2 \mp 2\sqrt{\frac{4a+b^5}{5b}}}\right)$$

**200.** 1. Given $\left.\begin{array}{ll} x^3 + y^3 = 9 & (1), \\ \text{and} \quad xy = 2 & (2), \end{array}\right\}$ to find $x$ and $y$.

From (2) $x^3y^3 = 8$, hence $x^3 = \dfrac{8}{y^3}$.

Then $\dfrac{8}{y^3} + y^3 = 9$.

Hence $x = 2$ or $1$, and $y = 1$ or $2$.

In the same way solve the equations—

| (2.) | (3.) | (4.) | (5.) |
|---|---|---|---|
| $(x^3+y^3=351.)$ | $(x^4-y^4=240.)$ | $(x^5+y^5=1267.)$ | $(x^4+y^4=17.)$ |
| $(xy=14.)$ | $(xy=8.)$ | $(xy=12.)$ | $(xy=2.)$ |

*Answers.*

| | | | |
|---|---|---|---|
| $x = 2$ or $7$. | $x = \pm 4$ or $\pm 2\sqrt[4]{-1}$. | $x = 4$ or $3$. | $x = 2$ or $1$. |
| $y = 7$ or $2$. | $y = \pm 2$ or $\pm 2\sqrt[4]{-1}$. | $y = 3$ or $4$. | $y = 1$ or $2$. |

**201.** In an equation in which the terms are homogeneous, we may, with great advantage, introduce an auxiliary unknown quantity, by letting $x = my$. The value of $m$ can easily be found, and from this $x$ and $y$. Thus,

1. Given $\left.\begin{array}{l} x(x+y) = 24 \text{ (1)}, \\ y(x-y) = 4 \text{ (2)}, \end{array}\right\}$ to find $x$ and $y$.

These equations may be written thus, by multiplying (2) by 6:

$$x^2 + xy = 24, \text{ and } 6xy - 6y^2 = 24.$$

$$\therefore \quad x^2 + xy = 6xy - 6y^2, \text{ or } x^2 = 5xy - 6y^2.$$

Now let $x = my$, whence $x^2 = m^2y^2$, and we have, by substitution in the last equation,

$$m^2y^2 = 5my^2 - 6y^2.$$

Divide this by $y^2$, and we have

$$m^2 - 5m = -6.$$

Hence $$m = 3 \text{ or } 2.$$

$$\therefore \quad x = 3y \text{ or } x = 2y.$$

Substitute the last value in (1), and we have

$$4y^2 + 2y^2 = 24.$$

Hence $$y = \pm 2, \text{ and } x = \pm 4.$$

Substitute the first value of $x$ in (1), and we have

$$4y^2 + 3y^2 = 24.$$

Hence $$y = \pm \sqrt{2}, \text{ and } x = \pm 3\sqrt{2}.$$

In the same way solve the equations—

2. $x(x+y) = 77$, and $y(x-y) = 12$.

*Ans.* $x = 7$ or $\frac{11}{2}\sqrt{2}$, $y = 4$ or $\frac{3}{2}\sqrt{2}$.

3. $x^2y + xy^2 = 6$, and $x^3 + xy^2 = 10$.

*Ans.* $x = 2$ or $1$, $y = 1$ or $-3$.

4. $x^2 + xy = 12$, and $xy - 2y^2 = 1$.

*Ans.* $x = \pm 3$ or $\frac{4}{3}\sqrt{6}$, $y = \pm 1$ or $\frac{1}{6}\sqrt{6}$.

5. $x^2y^2 + y^4 = 5$, and $x^4 + x^2y^2 = 20$.

*Ans.* $x = \pm 2$ or $\pm 2\sqrt{-1}$, $y = \pm 1$ or $\pm\sqrt{-1}$.

6 $x^2y^2 + x^4 = 20y^4$, and $x^2 + y^2 = 45$.

*Ans.* $x = \pm 6$ or $\mp \frac{15}{2}$, $y = \pm 3$ or $\pm \frac{3}{2}\sqrt{-5}$.

7. $x^2 + y^2 = 61$, and $x^2 - xy = 6$. *Ans.* $x = 6$, $y = 5$.

**202.** Sometimes we may introduce *two* auxiliary unknown quantities, one of which represents the *sum*, and the other the *product*, of $x$ and $y$. Thus,

1. Given $x^2y + xy^2 = 6$ (1), and $x^3 + y^3 = 9$ (2), to find $x$ and $y$

These equations may easily be written as follows:—

$xy(x + y) = 6$ (3), and $(x + y)^3 - 3xy(x + y) = 9$, (4).

Now let $x + y = a$, and $xy = b$, and the equations become

$$ab = 6 \text{ (5), and } a^3 - 3ab = 9, \text{ (6)}.$$

$$\therefore \quad a^3 = 27 \text{ or } a = 3, \text{ and } b = 2.$$

Hence $x + y = 3$, and $xy = 2$, from which we have

$$x = 2, \text{ and } y = 1.$$

2. Given $x^2 + y^2 - xy = 7$, and $x^3 + y^3 = 35$, to find $x$ and $y$.

*Ans.* $x = 3$, and $y = 2$.

3. Given $x^2 + y^2 + xy = 28$, and $x^3 - y^3 = 56$, to find $x$ and $y$.

*Ans.* $x = 4$, and $y = 2$.

4. Given $x^3 + y^3 - 2x^2y - 2xy^2 + \dfrac{5x^2y^2}{x + y} = 15\frac{1}{4}$, and $x^5 + y^5 = 244$, to find $x$ and $y$. All the values of $x$ and $y$ in these equations are as follows—

$x = 3$, and $y = 1$. $x = 2 \pm 3\sqrt{-1}$, and $y = 2 \mp 3\sqrt{-1}$.

$x = 1$, and $y = 3$.

$x = 2\left(-1 \pm \sqrt{-(1 + \frac{1}{20}\sqrt{15}}\right)$, and $y = 2\left(-1 \mp \sqrt{-(1 + \frac{1}{20}\sqrt{15}}\right)$

$x = 2\left(-1 \pm \sqrt{-(1 - \frac{1}{20}\sqrt{15}}\right)$, and $y = 2\left(-1 \mp \sqrt{-(1 - \frac{1}{20}\sqrt{15}}\right)$

**203.** Sometimes it is of much advantage to introduce two auxiliary quantities, one of which represents the *sum*, and the other the *difference*, of $x$ and $y$. Thus,

1. Given $x^3 - y^3 - xy^2 + x^2y = 25$ (1), and $x^3 + y^3 - xy^2 - x^2y = 5$ (2).

These equations are easily transformed into

(*vide* § **132** (2), ex. 7)

$(x - y)(x + y)^2 = 25$ (3), and $(x + y)(x - y)^2 = 5$ (4).

Now, let $x + y = a$, and $x - y = b$. By the substitution of these values in (3) and (4), we have

$$a^2b = 25, \text{ and } ab^2 = 5.$$

By the multiplication of which, we have

$$a^3b^3 = 125, \text{ or } ab = 5.$$

Hence $\quad a = 5$, and $b = 1$.

$\therefore \quad x + y = 5$, and $x - y = 1$.

From which $\quad x = 3$, and $y = 2$.

2. Given $x^3 + 3x^2(y - 1) + 3y^2(x + 1) + y^3 = 80$, and $x^2 + x(2y + 3) = 16 - y(y + 3)$, to find $x$ and $y$.

*Ans.* $x = 4$, $y = 1$.

3. Given $x^4 - y^4 + 2x^3y - 2xy^3 = 27$, and $x^2 - y^2 = 3$, to find $x$ and $y$. *Ans.* $x = \pm 2$, $y = \pm 1$.

**204.** If the preceding sections, commencing at § **182**, have been studied with sufficient care, the student will easily overcome all the difficulties attending equations of the kind we have been examining. We will finish this subject by adding a few

EQUATIONS CONTAINING THREE UNKNOWN QUANTITIES.

1. Given
$$\left.\begin{array}{rl} x + y + z = 6 & (1), \\ x^2 + y^2 + z^2 = 14 & (2), \\ \text{and} \quad x^2 + z^2 = 10 & (3), \end{array}\right\} \text{ to find } x, y, \text{ and } z.$$

*Ans.* $x = 1$, $y = 2$, $z = 3$.

2. Given
$$\left.\begin{array}{rl} 4x^2 + 4y^2 = 2z^2 + 2a^2 & (1), \\ 4y^2 + 4z^2 = 2x^2 + 2b^2 & (2), \\ 4x^2 + 4z^2 = 2y^2 + 2c^2 & (3), \end{array}\right\} \text{ to find } x, y, \text{ and } z.$$

*Ans.* $x = \pm \frac{1}{3}\sqrt{2a^2 + 2c^2 - b^2}$, $y = \pm \frac{1}{3}\sqrt{2b^2 + 2a^2 - c^2}$
$z = \pm \frac{1}{3}\sqrt{2c^2 + 2b^2 - a^2}$.

3. Given $\left.\begin{array}{ll} xy + z = 5 & (1), \\ xyz + z^2 = 15 & (2), \\ xy^2 + x^2y - 2x + 2z = 8 & (3), \end{array}\right\}$ to find $x$, $y$, and $z$.

*Ans.* $x = 2$, $y = 1$, $z = 3$.

## PROBLEMS.

### INVOLVING TWO UNKNOWN QUANTITIES.

**205.** 1. The sum of two numbers is 100, and the difference of the square roots of the numbers is 2. What are the numbers?

Let $x^2 =$ one, and $y^2 =$ the other number.

Then $x^2 + y^2 = 100$, and $x - y = 2$. Hence,

$$x^2 = 64, \text{ and } y^2 = 36.$$

2. The property of A and B together amounts to \$13,000, and each receives the same income. But if A should let his money at B's rate per cent., his income would be \$360, while B's income at A's rate per cent. would be \$490. What is the property of each?

Let $x =$ A's rate per cent., and $y =$ B's.

Then $\frac{36000}{y} =$ A's property, and $\frac{49000}{x} =$ B's.

Therefore, by the question,

$$\frac{36000}{y} + \frac{49000}{x} = 13{,}000. \quad (1)$$

and $$\frac{36000x}{y} = \frac{49000y}{x}. \quad (2)$$

$$36x + 49y = 13xy. \quad (3) = (1) \text{ reduced.}$$

$$6x = 7y. \quad (4) = (2) \text{ reduced.}$$

$$42y + 49y = \frac{91y^2}{6}. \quad (5) = (3) \text{ combined with } (4).$$

$y = 6$, and $x = 7$.

Hence $$\frac{36000}{6} = \$6000, \text{ and } \frac{49000}{7} = \$7000.$$

3. The sum of two numbers is 24, and their product is 35 times their difference. What are the numbers?

Let $x =$ the greater, and $y =$ the less number.

Then $x + y = 24$, and $xy = 35(x - y)$.

Find the value of $x$ in the first equation, and substitute it in the other. *Ans.* 14 and 12.

4. A number divided by the product of its digits gives a quotient of $2\frac{1}{3}$. If 18 be added to the number, the digits are inverted. What is the number? The equations are $\frac{10x + y}{xy} = 2\frac{1}{3}$, and $10x + y + 18 = 10y + x$. *Ans.* 35.

5. The sum of the digits of a certain number is 10, and if the product of the digits be increased by 40, the sum is the number inverted. What is the number? *Ans.* 46.

6. The sum of two numbers is $7\frac{1}{2}$, and the sum of the third powers is $343\frac{1}{8}$. What are the numbers? *Ans.* 7 and $\frac{1}{2}$.

7. The sum of two numbers is 47, and their product is 546. What is the sum of their squares? *Ans.* 1117.

8. The sum of two numbers is 20, and the product is 99. What is the sum of their cubes? (*Vide* **132** (2).) *Ans.* 2060.

9. The sum of two numbers is 8, and the product is 15. What is the sum of their fourth powers? *Ans.* 706.

10. The sum, product, and difference of the squares of two numbers are all equal. What are the numbers?

*Ans.* $\frac{1}{2}(3 \pm \sqrt{5})$ and $\frac{1}{2}(1 \pm \sqrt{5})$.

11. The sum of the squares, the product of the squares, and the difference of the fourth powers of two numbers are all equal. What are the numbers? *Ans.* 1.27203, and 1.61803.

12. The sum of the fourth powers of two numbers is $a$, and the product $b$. What are the numbers?

*Ans.* $\sqrt[4]{\frac{1}{2}(a \pm \sqrt{a^2 - 4b^4}}$, $\sqrt[4]{\frac{1}{2}(a \mp \sqrt{a^2 - 4b^4}}$

13. Divide 60 into two parts so that the product of the parts shall be to the difference of their squares as 2 to 3.

*Ans.* 40 and 20.

14. There are two numbers whose product is 77, and the difference of their squares is to the square of the difference as 9 to 2. What are the numbers? *Ans.* 11 and 7.

15. The product of two numbers is 48, and the difference of their cubes is to the cube of the difference as 37 to 1. What are the numbers? *Ans.* 8 and 6.

16. The difference of the fourth powers of two numbers divided by the difference of the numbers is 2336, and the product of the difference of their squares by the difference of the numbers is 576. What are the numbers? *Ans.* 11 and 5.

17. The product of two numbers is 320, and the difference of their cubes is equal to 61 times the cube of their difference. What are the numbers? *Ans.* 20 and 16.

18. Divide a number $a$ into two parts, so that the greater part may be a mean proportional between the whole number and the less part. Let $x$ = the greater part, and $y$ = the less.

Then $x + y = a$, and $a : x :: x : y$.

*Ans.* $\frac{a}{2}(3 - \sqrt{5})$, and $\frac{a}{2}(-1 + \sqrt{5}.)$

If $a = 20$, then the numbers are 12.36 and 7.64.

19. The sum of two numbers is $a$, and the sum of their reciprocals is $b$. What are the numbers?

*Ans.* $\frac{a}{2} + \sqrt{\frac{a^2}{4} - \frac{a}{b}}$, and $\frac{a}{2} - \sqrt{\frac{a^2}{4} - \frac{a}{b}}$.

20. The sum of the squares of two numbers is $a$, and the sum of the reciprocals of the numbers is $b$. What are the sum and product of the numbers?

*Ans.* Sum, $\frac{1}{b}(1 \pm \sqrt{1 + ab^2})$; product, $\frac{1}{b^2}(1 \pm \sqrt{1 + ab^2})$.

If $a = 5$ and $b = 1\frac{1}{2}$, then the numbers are 1 and 2.

21. A merchant bought 54 gallons of Madeira wine, and a certain quantity of Teneriffe. For the former he gave half as many shillings by the gallon as there were gallons of Teneriffe, and for the latter 4 shillings less by the gallon. He sold the mixture at 10 shillings by the gallon, and lost £28 10*s*. by his bargain. Required the price of the Madeira and the quantity of Teneriffe.

*Ans.* Madeira, 18 shillings; Teneriffe, 36 gallons.

22. The side of one square garden exceeds the side of another by 5 rods, and both gardens contain 1025 square rods. What is a side of each? *Ans.* 20 and 25.

23. A farmer has a field 16 rods long and 12 rods wide. He wishes to enlarge the field so that it may contain twice as much area, and not change the proportion of the sides. What will be the sides of the field? *Ans.* Length, $16\sqrt{2}$; breadth, $12\sqrt{2}$.

24. A rectangular grass-plat has its sides in the ratio of 4 to 3. A walk outside the plat, 6 feet wide, contains $\frac{1}{12}$ as much ground as the plat itself. What is the length and breadth of the plat? *Ans.* Length, 342.72; breadth, 257.04.

25. A grocer sold 80 pounds of mace and 100 pounds of cloves for £65, and finds that he has sold 60 pounds more of cloves for £20 than of mace for £10. What was the price of each per pound? *Ans.* 10 shillings and 5 shillings.

26. Find two numbers whose sum multiplied by the second is 84, and whose difference multiplied by the first is 16.

*Ans.* $\pm 8$ and $\pm 6$, or $\mp \sqrt{2}$ and $\pm 7\sqrt{2}$

27. The square of the sum of the squares of two numbers is 169, and the product of the squares is 36. What are the numbers? *Ans.* $\pm 3$ and $\pm 2$, or $\pm 3\sqrt{-1}$ and $\pm 2\sqrt{-1}$.

28. The difference of the fourth powers of two numbers, multiplied by the product of the squares, is 147,600. The sum of

the squares multiplied by the product of the numbers is 820 What are the numbers?

*Ans.* $\pm 5$ and $\pm 4$, or $\pm 5\sqrt{-1}$ and $\pm 4\sqrt{-1}$.

29. A and B bought a farm containing $a$ acres, each paying $m$ dollars. A paid $b$ dollars per acre more than B, in consideration of taking his share from the best portion. What does each one take, and at what price per acre?

*Ans.* A takes $\dfrac{2am}{2m + ab + \sqrt{4m^2 + a^2b^2}}$ at $\dfrac{2m + ab + \sqrt{4m^2 + a^2b^2}}{2a}$.

B takes $\dfrac{2am}{2m - ab + \sqrt{4m^2 + a^2b^2}}$ at $\dfrac{2m - ab + \sqrt{4m^2 + a^2b^2}}{2a}$.

If $m = a$, then A takes $\dfrac{2a}{2 + b + \sqrt{4 + b^2}}$ at $\dfrac{2 + b + \sqrt{4 + b^2}}{2}$,

and B takes $\dfrac{2a}{2 - b + \sqrt{4 + b^2}}$ at $\dfrac{2 - b + \sqrt{4 + b^2}}{2}$,

In this case—that is, when each pays as many dollars as there are acres—the price per acre does not depend upon the number of acres purchased.

If $m = a = 200$ and $b = \frac{3}{4}$ or \$.75, then A takes 81.867 acres at \$2.443, and B takes 118.133 acres at \$1.693.

If $m = a = 200$ and $b =$ \$2, then A takes 58.579 at \$3.41421 per acre, and B takes 141.421 at \$1.41421 per acre.

If $m = a = 300$ and $b =$ \$1.50, then A takes 100 acres at \$3.00 per acre, and B takes 200 acres at \$1.50 per acre.

**206.** Every equation must be regarded as the algebraic conditions of some problem. If, therefore, on solving the equation the value of $x$ is imaginary, it is absolutely impossible to fulfil the conditions of the problem.

30. Divide 10 into two parts, so that the product shall be 26.

Let $x =$ one, and $y =$ the other part.

Then $x + y = 10$, and $xy = 26$.

Hence $x - y = 2\sqrt{-1}$, and $x = 5 + \sqrt{-1}$, $y = 5 - \sqrt{-1}$

Therefore it is impossible to divide 10 into two parts so that the product shall be 26. This is readily seen on trial, thus:

$$10 = 9 + 1, \text{ and } 9 \times 1 = 9.$$
$$10 = 8 + 2, \text{ and } 8 \times 2 = 16.$$
$$10 = 7 + 3, \text{ and } 7 \times 3 = 21.$$
$$10 = 6 + 4, \text{ and } 6 \times 4 = 24.$$
$$10 = 5 + 5, \text{ and } 5 \times 5 = 25.$$

We see that *the product is greatest when the parts are equal.* That this is generally the case may readily be shown.

Let $x =$ one part, $y$ the other, $2s$ the sum, and $2d$ the difference.

Then $\qquad x + y = 2s$ and $x - y = 2d.$

Whence $\qquad x = s + d \quad (1).$

and $\qquad y = s - d \quad (2).$

The product of which is $xy = s^2 - d^2. \quad (3).$

It is plain that the second member of (3) *increases* as $d$ *diminishes*, and therefore that it is the greatest when $d = 0$, i.e. *when there is no difference between the parts.*

31. Divide 20 into two parts, so that the product shall be 150.

*Ans.* $x = 10 + 5\sqrt{-2}$, $y = 10 - 5\sqrt{-2}$.

32. The sum of two numbers is 1, and the sum of their reciprocals 2. What are the numbers? (*Vide* **167**, ex. 18.)

*Ans.* $\frac{1}{2}(1 + \sqrt{-1})$, and $\frac{1}{2}(1 - \sqrt{-1})$.

33. If 4 is added to a certain number, and the sum is divided by the number itself, the quotient is the same as that obtained by dividing three times the number by the number diminished by 4. What is the number? *Ans.* $\pm\sqrt{-2}$.

## GENERAL PROPERTIES OF EQUATIONS OF THE SECOND DEGREE.

*Definitions.*

**207.** 1. A *root* of an equation is a quantity which, being substituted for $x$ in the given equation, satisfies it.

2. An *imaginary root* is one involving the expression $\sqrt{-1}$.

3. A *real root* is one not involving an imaginary quantity.

4. *Equal roots* are where the roots are the same quantity.

## EQUATIONS OF THE SECOND DEGREE HAVE TWO ROOTS, AND ONLY TWO.

*Demonstration.*

**208.** Every equation of the second degree can be reduced to the form of

$$x^2 + 2px = q.$$

Add $p^2$ to both members, and we have

$$x^2 + 2px + p^2 = p^2 + q,$$

or
$$(x + p)^2 = p^2 + q;$$

and, by transposition,

$$(x + p)^2 - (p^2 + q) = 0,$$

or
$$(x + p + \sqrt{p^2 + q})\,(x + p - \sqrt{p^2 + q}) = 0.$$

(*Vide* **77.**)

Divide this equation first by one factor, and then by the other, and we have

$$\left.\begin{array}{l} x + p - \sqrt{p^2 + q} = 0 \;\therefore\; x = -p + \sqrt{p^2 + q}. \;(1) \\ \text{and} \\ x - p + \sqrt{p^2 + q} = 0 \;\therefore\; x = -v - \sqrt{p^2 + q}. \;(2) \end{array}\right\} \text{Q. E. D.}$$

**209.** If $p = 0$, then $x = \sqrt{q}$, and $x = -\sqrt{q}$, which are the roots of the incomplete equation $x^2 + 2 \times 0 . x = q$, or $x^2 = q$. (*Vide* **158** (3).)

**210.** *The algebraic sum of the two roots is equal to the coëfficient of the second term with its sign changed;* for

The roots are $x = -p + \sqrt{p^2 + q}$, and $x = -p - \sqrt{p^2 + q}$, the sum of which is $-2p$.

**211.** *The product of the two roots is equal to the second term with its sign changed;* for

The roots are $x = -p + \sqrt{p^2 + q}$, and $x = -p - \sqrt{p^2 + q}$, the product of which is $-q$.

### APPLICATION OF THESE PROPERTIES.

**212.** 1. What is the equation whose roots are 5 and $-9$?

By **210.** $2p = 4$, and by **211**, $q = 45$.

Therefore the equation is $x^2 + 4x = 45$.

2. What is the equation whose roots are 4 and 1?

*Ans.* $x^2 - 5x = -4$.

3. What is the equation whose roots are 9 and $-1$?

*Ans.* $x^2 - 8x = 9$.

4. What is the equation whose roots are $-2$ and $-2$?

*Ans.* $x^2 + 4x = -4$.

5. What is the equation whose roots are 1 and $-1\frac{7}{15}$?

*Ans.* $x^2 + \frac{7}{15} = 1\frac{7}{15}$.

6. What is the equation whose roots are 7 and $-8$?

*Ans.* $x^2 + x = 56$.

7. What is the equation whose roots are $-7$ and $-7$?

*Ans.* $x^2 + 14x = -49$.

8. What is the equation whose roots are 7 and 7?

*Ans.* $x^2 - 14x = -49$.

9. What is the equation whose roots are $a$ and $b$?

*Ans.* $x^2 - (a + b)\,x = -ab$.

10. What is the equation whose roots are $\frac{a\sqrt{b}+b\sqrt{a}}{a-b}$, and $\frac{a\sqrt{b}-b\sqrt{a}}{a-b}$?

*Ans.* $x^2 - \frac{2a\sqrt{b}\,x}{a-b} = \frac{-ab}{a-b}$.

**213.** Advantage may be taken of these general properties in solving any equation of the second degree.

1. Given $x^2 - 8x = -15$, to find the two roots.

Let $x =$ one root, and $y =$ the other.

Then $x + y = 8$, By 210.

and $xy = 15$. By 211.

Whence $x = 5$, and $y = 3$. *Vide* **182**, ex. 1.

2. Given $x^2 + 6x = 187$, to find the two roots.

Here $x + y = -6$,

and $xy = -187$.

Hence $x = 11$, and $y = -17$.

3. Given $x^2 + 4x = -4$, to find the roots.

Here $x + y = -4$,

and $xy = 4$.

Hence $x = -2$, and $y = -2$.

4. Given $3x^2 + \frac{7x}{5} = 4\frac{2}{5}$, to find the roots.

By reduction $x^2 + \frac{7x}{15} = \frac{22}{15}$.

Then $x + y = -\frac{7}{15}$,

and $xy = -\frac{22}{15}$.

Hence $x = 1$, and $y = -1\frac{7}{15}$.

5. Find the roots of $3x^2 - 2x = 8$. (*Vide* **176**, ex. 4.)

*Ans.* 2 and $-1\frac{1}{3}$.

6. Find the roots of $\frac{x+3}{x} + \frac{7x}{x+3} = 5\frac{3}{4}$. *Ans.* 4 and 1.

7. Find the roots of $x^2 - 20x = -104$.

*Ans.* $10 + 2\sqrt{-1}$ and $10 - 2\sqrt{-1}$

8. Find the roots of $x^2 - (a - 1)x = a$. *Ans.* $a$ and $-1$.

9. Find the roots of $\dfrac{x + a}{x - a} - b = \dfrac{a - x}{a + x}$.

*Ans.* (*Vide* **179**, ex. 6.)

10. Find the roots of $x^2 - 10x = -26$.

*Vide* **176**, ex. 44, and **206**, ex. 30.

**214.** Every equation of the second degree, as we have stated, may be reduced to the form $x^2 + 2px = q$ (1), the signs not being considered.

This form is obtained when $p$ and $q$ are *both positive.*

If $p$ is *negative* and $q$ *positive,* we have

$$x^2 - 2px = q \qquad (2)$$

If $p$ is *positive* and $q$ *negative,* we have

$$x^2 + 2px = -q \qquad (3)$$

If $p$ is *negative* and $q$ *negative,* we have

$$x^2 - 2px = -q \qquad (4)$$

And these are all the combinations that can be made in the signs; for if $x^2$ is negative, we may multiply, or divide, the whole equation by $-1$, and it will be found in one of the above forms. (*Vide* **176**, 5.)

The roots of these equations are respectively (*Vide* **171**, **174**)

$$x = -p \pm \sqrt{p^2 + q}. \qquad (1)$$

$$x = p \pm \sqrt{p^2 + q}. \qquad (2)$$

$$x = -p \pm \sqrt{p^2 - q}. \qquad (3)$$

$$x = p \pm \sqrt{p^2 - q}. \qquad (4)$$

From these roots we easily deduce the following facts:—

1. The roots of (1) and (2) are *always real.*

2. $\sqrt{p^2 + q} > p$. $\therefore$ The first root of (1) and (2) is *positive,* the second *negative.*

3. The negative root of (1) is numerically the larger.

4. The positive root of (2) is numerically the larger.

5. If $q < p^2$, the roots of (3) and (4) are *both real.*

6. If $q < p^2$, both roots of (3) are *negative,* and both roots of (4) are *positive.*

7 If $q = p^2$, the roots of (3) and (4) are *equal.*

8. If $q > p^2$, the roots of (3) and (4) are *imaginary.* (*Vide* **206.**)

9. If $p = 0$, the roots of (1) and (2) are numerically *equal,* but of contrary signs.

10. If $p = 0$, the roots of (3) and (4) are *imaginary* and *equal,* but of contrary signs.

11. If $q = 0$, the first root of (1) and (3) is 0, the second $-2p$.

12. If $q = 0$, the first root of (2) and (4) is $2p$, the second 0.

13. If $p = 0$, and $q = 0$, the roots of (1), (2), (3), and (4), are all 0.

## RATIO AND PROPORTION.

**215.** *Ratio is the quotient which is obtained by dividing one quantity by another of the same kind.* Thus,

The ratio of $a$ to $b$ is $\frac{b}{a}$, commonly expressed by $a : b$.

1. The two quantities forming a ratio are together called *terms.*

2. The first term alone is called the *antecedent.*

3. The second term alone is called the *consequent.* Thus,

$a : b$ are the *terms,* $a$ is the *antecedent,* and $b$ the *consequent.*

**216.** *A proportion* is an equality of ratios. Thus,

$$\frac{b}{a} = \frac{d}{c}, \text{ commonly written } a : b :: c : d.$$

1. The *first* and *last* terms of a proportion are called *extremes.*

2. The *second* and *third* terms are called *means.*

3. The first and second terms form the *first couplet.*

4. The third and fourth terms form the *second couplet.*

5. The last term is a *fourth proportional* to the other three. Thus,

$a$ and $d$ are *extremes,* $b$ and $c$ are *means,* $a$ and $b$ the *first couplet,* and $b$ and $d$ the *second couplet.*

6. If the means of a proportion are the *same quantity,* that quantity is called a *mean proportional* between the other two; and the last term is a *third proportional* to the first term and one of the means. Thus, in the proportion

$$a : b :: b : c,$$

$b$ is a mean proportional between $a$ and $c$, and $c$ is a fourth proportional to $a$ and $b$.

7. A continued proportion is one in which *several* ratios are equal. Thus,

$$a : b :: c : d :: m : n :: p : q, \text{ \&c.}$$

8. Three or four quantities are in *harmonical proportion* when the first is to the last, as the difference between the first two is to the difference between the last two. Thus,

$a, b, c,$ are in harmonical proportion when $a : c :: a - b : b - c.$
$a, b, c,$ and $d,$ " " " " $a : d :: a - b : c - d.$

**217.** Quantities are in proportion by *alternation,* when antecedent is compared with antecedent and consequent with consequent.

**218.** Quantities are in proportion by inversion, when antecedents are made consequents and consequents are made antecedents.

**219.** Quantities are in proportion by *composition*, when the sum of antecedent and consequent is compared with either antecedent or consequent.

**220.** Quantities are in proportion by *division*, when the difference of antecedent and consequent is compared with either antecedent or consequent.

**221.** Two varying quantities are *reciprocally* or *inversely* proportional when one is increased as many times as the other is diminished. Thus,

$x \times y = 2x \times \frac{y}{2} = 3x \times \frac{y}{3} = mx \times \frac{y}{m} = xy$, the product being *fixed*.

**222.** Equimultiples of two quantities are the results obtained by multiplying both by the same quantity. Thus,

$ma$ and $mb$ are equimultiples of $a$ and $b$.

**223.** Proposition I. *If four quantities are in proportion, the product of the extremes is equal to the product of the means.*

For, since $a : b :: c : d$, we have $\frac{b}{a} = \frac{d}{c}$. (1)

Clear (1) of fractions, and we have $ad = bc$. (2)

Cor 1. If $b = c$, then (2) becomes $ad = b^2$. (3)

That is, *The product of the extremes is equal to the square of the means.*

Cor. 2. If both members of (2) be divided by $ac$, we have

$$\frac{d}{c} = \frac{b}{a}, \text{ i.e. } \frac{b}{a} = \frac{d}{c}, \text{ or } a : b :: c : d. \qquad (12)$$

That is, *If the product of two quantities be equal to the product of two other quantities, the first two may be made the extremes, and the second two the means, of a proportion.*

PROP. II. *If four quantities are in proportion, they will be in proportion by alternation.*

For, since $a : b :: c : d$, we have $ad = bc$. (2) (Prop. 1.)

Dividing both members of (2) by $dc$, we have—

$$\frac{ad}{dc} = \frac{bc}{dc}, \text{ i.e. } \frac{a}{c} = \frac{b}{d}, \text{ whence } a : c :: b : d. \quad (10)$$

PROP. III. *If four quantities are in proportion, they will be in proportion by inversion.*

For, since $a : b :: c : d$, we have $ad = bc$, or $\frac{1}{bc} = \frac{1}{ad}$. (4)

Multiply both members of (4) by $ac$, and we have

$$\frac{ac}{bc} = \frac{ac}{ad}, \text{ i.e. } \frac{a}{b} = \frac{c}{d}, \text{ whence } b : a :: d : c. \quad (11)$$

PROP. IV. *If four quantities are in proportion, they will be in proportion by composition or division.*

For, since $a : b :: c : d$, we have $ad = bc$. (2)

Add or subtract $bd$ according to AX. I., and we have—

$$ad \pm bd = bc \pm bd, \text{ or } (a \pm b)\,d = (c \pm d)\,b, \text{ whence}$$
$$a \pm b : b :: c \pm d : d. \quad (5) \quad \text{(Prop. 1, Cor. 2, and Prop. 3.)}$$

From (2) we also have $ac \pm ad = ac \pm bc$, or $(c \pm d)\,a = (a \pm b)\,c$, whence

$$a \pm b : a :: c \pm d : d. \quad (6) \quad \text{(Prop. 1, Cor. 2).}$$

PROP. V. *If four quantities are in proportion, the sum of the first and second will be to their difference, as the sum of the third and fourth is to their difference.*

For, by (5) and Prop. 2, we have—

$$a + b : c + d :: b : d, \text{ i.e. } \frac{c + d}{a + b} = \frac{d}{b}.$$

and

$$a - b : c - d :: b : d, \text{ i.e. } \frac{c - d}{a - b} = \frac{d}{b}.$$

whence

$$\frac{c + d}{a + b} = \frac{c - d}{a - b}: \text{ that is, } a + b : a - b :: c + d : c - d. \quad (7) \quad \text{(Prop. 2.)}$$

PROP. VI. *Equimultiples of two quantities are proportional to the quantities themselves.*

For $$\frac{mb}{ma} = \frac{b}{a}, \text{ or } ma : mb :: a : b. \tag{8}$$

COR. Since $$\frac{mb}{ma} = \frac{nb}{na}, \text{ we have } ma : mb :: na : nb. \tag{9}$$

PROP. VII. *If four quantities are in proportion, the like powers or like roots will be in proportion.*

For, since $a : b :: c : d$, we have $ad = bc$, or $a^m d^m = b^m c^m$, in which $m$ is a whole number or a fraction. Restore the proportion, and we have

$$a^m : b^m :: c^m : d^m. \tag{13}$$

PROP. VIII. *If two sets of quantities are in proportion, the products of the corresponding terms will be in proportion.*

For, since $a : b :: c : d$, we have $ad = bc$;

and since $m : n :: p : q$, we have $mq = np$;

whence $am \times dq = bn \times cp$, or $am : bn :: cp : dq$. (14)

PROP. IX. *In a continued proportion, the sum of all the antecedents is to the sum of all the consequents as any one antecedent is to its consequent.* (*Vide* § **2**, def. 7.)

For, since $a : b :: c : d$, we have $ad = bc$;

and since $a : b :: m : n$, we have $an = bm$;

and since $a : b :: a : b$, we have $ab = ab$. The sum of these equations is

$$(b+d+n)\,a = (a+c+m)\,b, \text{ or } a+c+m : b+d+n :: a : b. \tag{15}$$

PROP. X. *If the first couplets of two proportions are the same, the second couplets will form a proportion.*

For, since $a : b :: c : d$, and $a : b :: e : f$,

we have $\frac{b}{a} = \frac{d}{c}$, and $\frac{b}{a} = \frac{f}{e}$.

$$\therefore \frac{d}{c} = \frac{f}{e}, \text{ whence } c : d :: e : f. \tag{16}$$

Cor. By alternation, *If the antecedents of two proportions are the same, the consequents will be proportional.*

ILLUSTRATION OF THE PRECEDING PROPOSITIONS.

$4 : 3 :: 20 : 15$. By Prop. 1st, $4 \times 15 = 3 \times 20$, i.e. $60 = 60$

$4 : 3 :: 20 : 15$. " 2nd, $4 : 20 :: 3 : 15$.

$4 : 3 :: 20 : 15$. " 3d, $3 : 4 :: 15 : 20$.

$4 : 3 :: 20 : 15$. " 4th, $4 \pm 3 : 4 :: 20 \pm 15 : 20$,

i.e. $\left\{\begin{array}{l} 7 : 4 :: 35 : 20, \\ 1 : 4 :: \phantom{3}5 : 20. \end{array}\right\}$

$4 : 3 :: 20 : 15$. " 5th, $4 + 3 : 4 - 3 :: 20 + 15 : 20 - 15$,

i.e. $7 : 1 :. 35 : 5$.

$4 : 3 :: 20 : 15$. " 7th, $4^2 : 3^2 :: 20^2 : 25^2$,

i.e. $16 : 9 :: 400 : 225$.

$4 : 3 :: 20 : 15 :: 44 : 33$. By Prop. 9th, $4 + 20 + 44 : 3 + 15 + 33 :: 4 : 3$,

i.e. $68 : 51 :: 4 : 3$.

$\left.\begin{array}{l} 4 : 3 :: 20 : 15, \\ 5 : 8 :: 10 : 16. \end{array}\right\}$ By Prop. 8th, $20 : 24 :: 200 : 240$.

$\left.\begin{array}{l} 4 : 3 :: 20 : 15, \\ 4 : 3 :: 44 : 33. \end{array}\right\}$ By Prop. 10th, $20 : 15 :: 44 : 33$.

## PROBLEMS.

**224.** 1. Any three terms of a proportion being given, to find the other term.

In the proportion $a : b :: c : d$,

let $x$ take the place of $a$, $b$, $c$, and $d$, in succession. In each case we are to find the value of $x$.

$x : b :: c : d$, whence $dx = bc$, i.e. $x = \dfrac{bc}{d}$. (1)

$a : x :: c : d$, " $cx = ad$, i.e. $x = \dfrac{ad}{c}$. (2)

$a : b :: x : d$, " $bx = ad$, i.e. $x = \dfrac{ad}{b}$. (3)

$a : b :: c : x$, " $ax = bc$, i.e. $x = \dfrac{bc}{a}$. (4)

(1) and (4) show that *either extreme is equal to the product of the means divided by the other extreme.*

(2) and (3) show that *either mean is equal to the product of the extremes divided by the other mean.*

2. Find the value of $x$ in the proportion $x : 2 :: 5 : 1$. *Ans.* 10.

3. Find the value of $x$ in the proportion $12 : x :: 4 : 7$. *Ans.* 21.

4. Find the value of $x$ in the proportion $8 : 5 :: x : 10$. *Ans.* 16.

5. Find the value of $x$ in the proportion $14 : 12 :: 7 : x$. *Ans.* 6.

6. To find a mean proportional between two quantities.

In the proportion $a : x :: x : d$, we have $x^2 = ad \therefore x = \sqrt{ad}$.

Hence, *the mean proportional between two quantities is the square root of the product of the quantities.*

7. Find the mean proportional between 9 and 4. *Ans.* $\sqrt{36} = 6$.

8. Given the proportions $\left\{\begin{array}{l} a : x :: y : b, \\ a : m :: p : b, \end{array}\right\}$ to find the relations of $x$, $y$, $m$, and $p$.

*Ans.* $x : m :: p : y$.

9. Given $x^2 : (14 - x)^2 :: 16 : 9$, to find $x$.

$x : 14 - x :: 4 : 3$. Prop. 8th.

$x : 14 :: 4 : 7$. Prop. 4th.

$7x = 56$. Prop. 1st.

$x = 8$.

10. Given $xy = 24$ and $x^3 - y^3 : (x-y)^3 :: 19 : 1$, to find $x$ and $y$.

We have

$x^3 - y^3 : x^3 - 3x^2y + 3xy^2 - y^3 :: 19 : 1$.

$3x^2y - 3xy^2 : (x - y)^3 :: 18 : 1$. Prop. 4.

$xy\,(x - y) : (x - y)^3 :: 6 : 1$. Dividing antecedents by 3.

$xy : (x - y)^2 :: 6 : 1$. Dividing 1st couplet by $(x-y)$.

$24 : (x - y)^2 :: 6 : 1$. Since $xy = 24$.

$4 : (x - y)^2 :: 1 : 1$.

Hence $x - y = 2$ and $xy = 24$, whence $x = 6$ and $y = 4$.

11. Given $(a + x)^2 : (a - x)^2 :: x + y : x - y$, to prove that

$$a : x :: \sqrt{2a - y} : \sqrt{y}.$$

| | |
|---|---|
| $a^2+2ax+x^2 : a^2-2ax+x^2 :: x+y : x-y.$ | By expanding. |
| $2a^2 + 2x^2 : 4ax :: 2x : 2y.$ | Prop. 5th. |
| $a^2 + x^2 : 2ax :: x : y.$ | Dividing by 2. |
| $a^2 + x^2 : 2a :: x^2 : y.$ | Transferring the factor $x$. |
| $a^2 + x^2 : x^2 :: 2a : y.$ | Prop. 2nd. |
| $a^2 : x^2 :: 2a - y : y.$ | Prop. 4th. |
| $a : x :: \sqrt{2a-y} : \sqrt{y}.$ | Prop. 7th. |

12. Given $xy = 135$ and $x^2 - y^2 : (x-y)^2 :: 4 : 1$, to find $x$ and $y$. *Ans.* $x = 15, y = 9.$

13. Given $\left\{\begin{array}{l} x - y : x + y :: 2 : 3, \\ x + y : \quad xy :: 3 : 5, \end{array}\right\}$ to find $x$ and $y$.

*Ans.* $x = 10, y = 2.$

14. Given $x + y = 24$ and $xy : x^2 + y^2 :: 3 : 10$, to find $x$ and $y$. *Ans.* $x = 18, y = 6.$

15. Given $x : y :: 3 : 2$ and $x + 6 : y - 6 :: 3 : 1$, to find $x$ and $y$ *Ans.* $x = 24, y = 16.$

16. Given $xy = 320$ and $x^3 - y^3 : (x-y)^3 :: 61 : 1$, to find $x$ and $y$. *Ans.* $x = 20, y = 16.$

## ARITHMETICAL PROGRESSION.

**225.** A *Series* is a succession of terms, each of which is derived from one or more of the preceding terms, by a *fixed law.*

1, 3, 5, 7, 9, &c.

is a series in which any term is derived from the preceding one by *adding* 2.

3, 6, 12, 24, 48, &c.

is a series in which any term is derived from the one preceding by *multiplying* by 2.

1, 3, 4, 7, 11, 18, &c.

is a series in which any term is found by *adding the two preceding* it, after the second term.

**226.** An *Arithmetical Progression* is a series whose law is that *any term is found by adding a constant quantity to the preceding term.*

The *common difference* is the constant quantity to be added.

The progression is *increasing* when the common difference is *positive.*

The progression is *decreasing* when the common difference is *negative.*

The *number of terms* of a series may be limited or infinite.

The *first term* is that with which the progression commences.

The *last term* is that with which the progression is supposed to terminate.

The *sum of the terms* is the amount of all the terms of the progression.

$$3, 7, 11, 15, 19,$$

is an increasing arithmetical progression, in which 3 is the *first term*, 4 is the *common difference*, 19 is the *last term*, 5 is the *number of terms*, and 55 is the *sum of the terms.*

$$19, 15, 11, 7, 3,$$

is a decreasing arithmetical progression, in which 19 is the *first term,* $-4$ is the *common difference,* 3 is the *last term,* 5 is the *number of terms*, and 55 is the *sum of the terms.*

$$4, 3, 2, 1, 0, -1, -2, -3, -4,$$

is a decreasing arithmetical progression, in which 4 is the *first term,* $-1$ is the *common difference,* $-4$ is the *last term,* 9 is the *number of terms*, and the *sum of the terms* is 0.

**227.** To find the last term, when the *first term*, *number of terms*, and the *common difference*, are known.

Let $l =$ *last term,* $a =$ *first term,* $n =$ *number of terms,* and $d =$ *common difference;* then the progression will be

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad a + 4d, \quad a + 5d, \text{ \&c.}$$

In which any of the numerical coëfficients is represented by $n-1$.

Therefore, $$l = a + (n-1)\,d. \qquad (1)$$

If $d$ is negative, then $$l = a - (n-1)\,d. \qquad (2)$$

2. To find the sum of the terms, when the first term, the *last term*, and the *number of terms*, are known.

Let $s$ = *sum of the terms*, $l$ = *last term*, $a$ = *first term*, and $n$ = *number of terms*; then, writing the progression in a *direct* and *reverse* order, we have the equations—

$$s = a + \overline{a+d} + \overline{a+2d} + \overline{a+3d} + \ldots\ldots l.$$
$$s = l + \overline{l-d} + \overline{l-2d} + \overline{l-3d} + \ldots\ldots a.$$

By addition, $2s = \overline{a+l} + \overline{a+l} + \overline{a+l} + \overline{a+l} + \ldots\ldots \overline{a+l}$, in which $a+l$ is taken as many times as is indicated by $n$, the number of terms.

Therefore, $$2s = (a+l)\,n, \text{ or } s = \frac{(a+l)\,n}{2}. \qquad (3)$$

From equations (1) and (3) the following table is easily made:—

| No. | Given. | Unknown. | Values of the Unknown Quantities. | |
|---|---|---|---|---|
| 1. | $a, d, n,$ | $l, s,$ | $l = a + (n-1)\,d;$ | $s = \frac{1}{2}n\,[2a + (n-1)d].$ |
| 2. | $a, d, l,$ | $n, s,$ | $n = \frac{1}{d}\,(l-a) + 1;$ | $s = \frac{1}{2d}(l+a)\,(l-a+d).$ |
| 3. | $a, d, s,$ | $n, l,$ | $n = \frac{d-2a \pm \sqrt{(d-2a)^2+8ds}}{2d};$ | $l = a + (n-1)\,d.$ |
| 4. | $a, n, l,$ | $s, d,$ | $s = \frac{1}{2}n\,(a+l);$ | $d = \frac{l-a}{n-1}.$ |
| 5. | $a, n, s,$ | $d, l,$ | $d = \frac{2\,(s-an)}{n\,(n-1)};$ | $e = \frac{2s}{n} - a.$ |
| 6. | $a, l, s,$ | $n, d,$ | $n = \frac{2s}{a+l};$ | $d = \frac{(l+a)\,(l-a)}{2s-(l+a)}.$ |
| 7. | $d, n, l,$ | $a, s,$ | $a = l - (n-1)\,d;$ | $s = \frac{1}{2}n\,[2l - (n-1)\,d].$ |
| 8. | $d, n, s,$ | $a, l,$ | $a = \frac{2s - n\,(n-1)\,d}{2n};$ | $l = \frac{2s + n\,(n-1)\,d}{2n}.$ |
| 9. | $d, l, s,$ | $n, a,$ | $n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}$ | $a = l - (n-1)d.$ |
| 10. | $n, l, s,$ | $a, d,$ | $a = \frac{2s}{n} - l;$ | $d = \frac{2\,(nl-s)}{n\,(n-1)}.$ |

EXAMPLES.

**228.** 1. If $a = 1$, $d = 2$, what is the sum of $n$ terms of the progression?

Here $s = \frac{1}{2}n\,[2a + (n-1)\,d]$, i.e. $\frac{1}{2}n\,[2 + (n-1)2]$, or $s = n^2$.

2. What is the sum of 25 terms of the progression 1, 3, 5, 7, 9, &c.? *Ans.* 625.

3. If $a = 1$ and $d = 1$, what is the sum of $n$ terms of the progression? *Ans.* $\frac{n\,(n+1)}{2}$.

4. How many strokes does the bell of a clock make in 12 hours? *Ans.* 78.

5. If $a = 2$, $l = 29$, and $d = 3$, find $n$ and $s$. *Ans.* $n = 10$, $s = 155$.

6. If $a = 5$, $d = 6$, and $s = 2945$, find $l$ and $n$. *Ans.* $n = 31$, $l = 185$.

7. If $a = 5$, $n = 31$, and $l = 185$, find $d$ and $s$. *Ans.* $d = 6$, $s = 2945$.

8. If $a = 1$, $n = 14$, and $s = 196$, find $d$ and $l$. *Ans.* $d = 2$, $l = 27$.

9. If $a = 1$, $l = 20$, and $s = 210$, find $n$ and $d$. *Ans.* $d = 1$, $n = 20$.

10. If $d = 3$, $n = 21$, and $l = 70$, find $a$ and $s$. *Ans.* $a = 10$, $s = 840$.

11. If $d = 3$, $n = 21$, and $s = 840$, find $a$ and $l$. *Ans.* $a = 10$, $l = 70$.

12. If $d = -4$, $l = 12$, and $s = 72$, find $a$ and $n$. *Ans.* $a = 24$, $n = 4$ or 9.

The series is 24, 20, 16, 12, or 24, 20, 16, 12, 8, 4, 0, $-4$, $-8$.

13. If $n = 21$, $l = 70$, and $s = 840$, find $d$ and $a$. *Ans.* $d = 3$, $a = 10$.

14. Insert four arithmetical means between 1 and 16.

Since the means are 4, the number of terms must be 6.

Hence the formula $d = \frac{l - a}{n - 1}$ becomes $d = \frac{16 - 1}{6 - 1} = 3$.

$\therefore$ The series is 1, 4, 7, 10, 13, 16.

**229.** 15. A starts from a certain place, travelling 1 mile the first day, 2 the second, 3 the third, and so on. At the end of 4 days, B starts from the same place, travelling uniformly at the rate of 9 miles a day. When will B overtake A?

Let $x =$ the time.

The distance travelled by A will be $\frac{x(x+1)}{2}$. (*Vide* ex. 3.)

The distance travelled by B will be $9(x-4)$.

Therefore, $\frac{x(x+1)}{2} = 9(x-4)$. *Ans.* $x = 8$, or 9.

In 8 days each will have travelled 36 miles. In 9 days, 45 miles.

**230.** 16. A starts from a certain place, travelling 1 mile the first day, 3 the second, 5 the third, and so on. At the end of two days, B starts from the same place, travelling uniformly at the rate of 9 miles a day. When will they be together?

*Ans.* In 3 days, and again in 6 days.

17. The sum of four numbers in arithmetical progression is 56. The sum of their squares is 864. What are the numbers?

Let the numbers be $x - y, x, x + y, x + 2y$

Then $4x + 2y = 56$,

and $4x^2 + 4xy + 6y^2 = 864$.

The numbers are 8, 12, 16, 20.

18. The sum of four numbers in arithmetical progression is 28. Their continued product is 585. What are the numbers?

Let the series be $x - 3y, x - y, x + y, x + 3y$.

Then $4x = 28$, and $x^4 - 10x^2y^2 + 9y^4 = 585$.

*Ans.* 1, 5, 9, 13.

**231.** A *Geometrical Progression* is a series whose law is, *that any term is found by multiplying the preceding term by a constant quantity.*

The *ratio* is the constant quantity used as a multiplier.

The progression is *increasing* when the ratio is *greater* than *unity.*

The progression is *decreasing* when the ratio is *less* than *unity.*

The *first term, last term, number of terms,* and *sum of the terms* are the same as in arithmetical progression.

$$1,\ 3,\ 9,\ 27,\ 81,$$

is a geometrical progression, increasing, whose *ratio* is 3, *first term* 1, *last term* 81, *number of terms* 5, and *sum of terms* 121.

$$4,\ 2,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},$$

is a decreasing geometrical progression, whose *ratio* is $\frac{1}{2}$, *first term* 4, *last term* $\frac{1}{8}$, *number of terms* 6, and *sum of terms* $7\frac{7}{8}$.

1. To find the *last term*, when the *first term, number of terms,* and *ratio* are known.

Let $l$ = last term, $n$ = number of terms, and $r$ = *ratio*, $a$ = *first term.* In this case the progression will be—

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ \&c.;$$

in which any exponent is represented by $n - 1$.

Therefore, $$l = ar^{n-1}. \qquad (1)$$

2. To find the *sum of the terms*, when the *first term, number of terms,* and *ratio* are given. If, in addition to the above notation, $s$ = sum of the terms, then

$$s = a + ar + ar^2 + \dots + ar^{n-1}.$$

Multiplying by $r$, $$rs = ar + ar^2 \quad + ar^{n-1} + ar^n.$$

whence $$rs - s = ar^n - a,$$

and $$s = \frac{ar^n - a}{r - 1} = \frac{a(r^n - 1)}{r - 1}. \qquad (2)$$

From (1) and (2) the following table is readily formed :—

| No. | Given. | Required. | Formulas. |
|---|---|---|---|
| 1. | $a, r, n,$ | | $l = ar^{n-1}.$ |
| 2. | $a, r, s,$ | $l,$ | $l = \dfrac{a + (r-1)}{r}.$ |
| 3. | $a, n, s,$ | | $l\,(s - l)^{n-1} - a\,(s - a)^{n-1} = 0.$ |
| 4. | $r, n, s,$ | | $l = \dfrac{(r-1)\,sr^{n-1}}{r^{n}-1}.$ |
| 5. | $a, r, n,$ | | $s = \dfrac{a\,(r^{n} - 1)}{r - 1}.$ |
| 6. | $a, r, l,$ | $s,$ | $s = \dfrac{rl - a}{r - 1}.$ |
| 7. | $a, n, l,$ | | $s = \dfrac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}.$ |
| 8. | $r, n, l,$ | | $s = \dfrac{l\,(r^{n} - 1)}{(r-1)\,r^{n-1}}.$ |
| 9. | $r, n, l,$ | | $a = \dfrac{l}{r^{n-1}}.$ |
| 10. | $r, n, s,$ | $a,$ | $a = \dfrac{(r-1)\,s}{r^{n} - 1}.$ |
| 11. | $r, l, s,$ | | $a = rl - (r-1)\,s.$ |
| 12. | $n, l, s,$ | | $a\,(s - a)^{n-1} - l\,(s - l)^{n-1} = 0.$ |
| 13. | $a, n, l,$ | | $r = \left(\dfrac{l}{a}\right)^{\frac{1}{n-1}}.$ |
| 14. | $a, n, s,$ | $r,$ | $ar^{n} - rs + s - a = 0.$ |
| 15. | $a, l, s,$ | | $(s - l)\,r^{n} - sr^{n-1} + l = 0. \quad r = \dfrac{s-a}{s-l}.$ |
| 16. | $n, l, s,$ | | $(s - l)\,r^{n} - sr^{n-1} + l = 0.$ |
| 17. | $a, r, l,$ | | $n = \dfrac{\log l - \log a}{\log r} + 1.$ |
| 18. | $a, r, s,$ | $n,$ | $n = \dfrac{\log\,[a + (r-1)\,s] - \log a}{\log r}.$ |
| 19. | $a, l, s,$ | | $n = \dfrac{\log l - \log a}{\log (s - a) - \log (s - l)} + 1.$ |
| 20. | $r, l, s,$ | | $n = \dfrac{\log l - \log\,[rl - (r-1)\,s]}{\log r} + 1.$ |

EXAMPLES.

1. If $a = 5$, $r = 10$, and $n = 7$, what is the sum of the series? *Ans.* 55,555,555.

2. If $a = 1$, $r = 2$, and $n = 7$, what is the last term? *Ans.* 64.

3. If $a = 1$, $r = 2$, and $s = 127$, what is the number of terms? *Ans.* 7.

4. If $a = 1$, $l = 27$, and $s = 40$, what is the number of terms? *Ans.* 4.

5 Insert 4 geometrical means between 2 and 486.

$$\text{Since } r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}, \text{ we have } r = \sqrt[5]{243} = 3.$$

$\therefore$ 2, 6, 18, 54, 162, 486, is the series.

6. Insert 5 means between 1 and $\frac{1}{64}$.

The series is, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$.

**232.** Formula (6) is $s = \dfrac{rl - a}{r - 1} = \dfrac{a - rl}{1 - r}$.

If $r$ is a proper fraction, the progression is decreasing; and if the series be carried to infinity, the last term becomes 0.

The formula will then become $s = \dfrac{a}{1 - r}$. (21)

EXAMPLES.

1. Find the sum of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$, &c. to $l = 0$.

Here $a = \frac{1}{2}$, $r = \frac{1}{2}$: hence $s = \dfrac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$. *Ans.* 1.

2. Find the sum of the series $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$, &c. to infinity.

$$s = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}. \qquad \textit{Ans. } \tfrac{1}{2}.$$

3. Find the sum of the series $1 + \frac{4}{5} + \frac{16}{25}$, &c. to infinity.

$$s = \frac{1}{1 - \frac{4}{5}} = 5. \qquad \textit{Ans. } 5.$$

4. Find the sum of the series $\frac{2}{3} + \frac{1}{2} + \frac{3}{8}$, &c.

$$s = \frac{\frac{2}{3}}{1 - \frac{3}{4}} = \frac{\frac{2}{3}}{\frac{1}{4}} = 2\tfrac{2}{3}. \qquad \textit{Ans. } 2\tfrac{2}{3}.$$

5. What is the sum of the series $1 + \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3}$, &c.?

$$s = \frac{1}{1 - \frac{1}{x}} = \frac{x}{x - 1}. \qquad \textit{Ans. } \frac{x}{x-1}.$$

6. What is the sum of the series $1 + \dfrac{1}{x+1} + \dfrac{1}{(x+1)^2} + \dfrac{1}{(x+1)^3}$, &c.?

$$s = \frac{1}{1 - \frac{1}{x+1}} = \frac{x+1}{x}.$$

7. What is the sum of the series $x + xy + xy^2 + xy^3$, &c. when $y < 1$? *Ans.* $\dfrac{x}{1-y}$.

PROBLEMS.

**233.** 1. The sum of three numbers in geometrical progression is 14, and the sum of their squares is 84. What are the numbers?

*Solution.*

Let the numbers be $x$, $xy$, and $xy^2$.

Then $\quad x + xy + xy^2 = 14, \qquad (1)$

and $\quad x^2 + x^2y^2 + x^2y^4 = 84. \qquad (2)$

Divide (2) by (1), and we have

$$x - xy + xy^2 = 6. \qquad (3)$$

Subtract (3) from (1), and we have

$$2xy = 8, \text{ or } xy = 4. \qquad (4)$$

For $x$ place $\dfrac{4}{y}$ in equation (1), and we have

$$\frac{4}{y} + 4 + 4y = 14. \qquad (5)$$

whence $\quad y = 2$, or $\frac{1}{2}$.

$\therefore \quad x = 2$, or $8$.

And the series is 2, 4, 8, or 8, 4, 2.

2. The sum of three numbers in geometrical progression is 21, and the sum of their reciprocals is $\frac{7}{12}$. What are the numbers?

$$x + xy + xy^2 = 21. \qquad (1)$$

$$\frac{1}{x} + \frac{1}{xy} + \frac{1}{xy^2} = \tfrac{7}{12}. \qquad (2)$$

whence $$1 + y + y^2 = \frac{21}{x}, \qquad (3)$$

and $$1 + y + y^2 = \frac{7xy^2}{12}. \qquad (4)$$

$\therefore$ $$\frac{21}{x} = \frac{7xy^2}{12}, \text{ whence } xy = 6.$$

Substitute the value of $xy$ in (4), and we have

$$1 + y + y^2 = \frac{42y}{12}.$$

whence $y = 2$ or $\frac{1}{2}$.

$\therefore$ $x = 3$ or $12$.

And the series is 3, 6, 12, or 12, 6, 3.

3. The product of three numbers in geometrical progression is 64, and the sum of their cubes is 584. What are the numbers?

The equations are $x \times xy \times xy^2 = 64$,

and $x^3 + x^3y^3 + x^3y^6 = 584$.

The numbers are 2, 4, 8.

4. The sum of the first and last of three numbers in geometrical progression is 52. The square of the mean is 100. What are the numbers? *Ans.* 2, 10, 50.

5. The sum of the first and second of four numbers in geometrical progression is 15. The sum of the third and fourth is 60. What are the numbers?

$$x + xy = 15, \quad \text{and} \quad xy^2 + xy^3 = 60.$$

*Ans.* 5, 10, 20, 40.

6. The sum of the extremes of four numbers in geometrical

progression is 84. The sum of the means is 36. What are the numbers?

$$xy + xy^2 = 36, \quad \text{and} \quad x + xy^3 = 84,$$

or, $$xy(1 + y) = 36, \quad \text{and} \quad x(1 + y^3) = 84.$$

whence $$\frac{1}{y}(1 - y + y^2) = \tfrac{7}{3}. \qquad y = 3, \text{ or } \tfrac{1}{3}.$$

The series is 3, 9, 27, 81.

7. The difference between the second and fourth of four numbers in geometrical progression is 24. The sum of the extremes is to the sum of the means as 7 to 3. What are the numbers?

*Ans.* 1, 3, 9, 27.

8. What is the third term of a harmonical proportion whose first and second terms are 12 and 15?

If $x$ represent the required term, we have

$$12 : x :: 15 - 12 : x - 15. \quad (\textit{Vide} \text{ Def. } \mathbf{8}.)$$

whence $$x = 20.$$

## INDETERMINATE COËFFICIENTS.

**234.** Every identical equation containing but one unknown quantity can be reduced to the form of—

$$p + qx + rx^2 +, \text{\&c.} = p^1 + q^1x + r^1x^2 +, \text{\&c.} \quad (1)$$

Or, by transposition, to the form—

$$(p - p^1) + (q - q^1)x + (r - r^1)x^2 +, \text{\&c.} = 0. \quad (2)$$

**235.** From the nature of an identical equation (*vide* **92**, 5), equations (1) and (2) must be true for all possible values of $x$; that is, we may take $x = \frac{0}{0} = 0, 1, 2, 3,$ &c.; and what is true of the coëfficients when $x$ equals any *one* of these values, is true when any *other* value is taken for $x$.

**236.** Because the coëfficients of the different powers of the unknown quantity in equation (2) are coëfficients of indeterminate quantities, they are called *Indeterminate Coëfficients.*

**237.** In any identical equation, containing but one unknown quantity, *the coëfficients of the like powers of this quantity in the two members are equal to each other.* For, if $x = 0$ in equation (1), the equation reduces to

$$p = p^1.$$

If now these quantities are cancelled, we have

$$qx + rx^2 +, \&c. = q^1x + r^1x^2 +, \&c.$$

Or, by dividing by $x$,

$$q + rx +, \&c. = q^1 + r^1x +, \&c.$$

Now make $x = 0$, and we have

$$q = q^1.$$

In the same way we may obtain

$$r = r^1.$$

**238.** To develope an expression by means of the *principle of indeterminate coëfficients.*

1. Assume the expression to be equal to a series of the form $p + qx + rx^2 + sx^3 +, \&c.$

2. Clear the equation of fractions, or raise it to the required power.

3. Equate the coëfficients of the like powers of the unknown quantity.

4. Find from these equations the values of $p, q, r, \&c.$

5. Substitute these values in the assumed development.

EXAMPLES.

1. Develope $\dfrac{1 + 2x}{1 - x - x^2}$ into a series.

*Operation.*

$$\frac{1 + 2x}{1 - x - x^2} = p + qx + rx^2 + sx^3 + tx^4, \&c.$$

Clear this of fractions, and we have

$$1 + 2x = p + \begin{array}{r|r|r|r|l} q & x + r & x^2 + s & x^3 + t & x^4, \text{ \&c.} \\ -p & -q & -r & -s & \\ & -p & -q & -r & \end{array}$$

Equating the like powers of $x$, we have

$$\begin{array}{ll} (1 - p)\,x^0 = 0, & \text{whence } p = 1. \\ 2 = -p + q, & \quad\text{"} \quad q = 3. \\ 0 = r - q - p, & \quad\text{"} \quad r = 4. \\ 0 = s - r - q, & \quad\text{"} \quad s = 7. \\ 0 = t - s - r, & \quad\text{"} \quad t = 11. \end{array}$$

Hence $\dfrac{1 + 2x}{1 - x - x^2} = 1 + 3x + 4x^2 + 7x^3 + 11x^4$, &c.

(*Vide* **70**, ex. 26.)

2. Develope $\dfrac{1 - x}{1 + x + x^2}$ into a series. (*Vide* **70**, ex. 25.)

3. Develope $\dfrac{1 + 2x + 3x^2}{1 - 4x}$ into a series. (*Vide* **70**, ex. 23.)

4. Develope $\dfrac{2 + 3x}{3 + 4x + 5x^2}$ into a series.

*Ans.* $\frac{2}{3} + \dfrac{x}{9} - \dfrac{34x^2}{27} + \dfrac{121x^3}{81}$, &c.

5. Develope $\sqrt{1 - x^2}$ into a series.

*Operation.*

$$\sqrt{1 - x^2} = p + qx + rx^2 + sx^3 + tx^4, \text{ \&c.}$$

Square both sides, and we have

$$1 - x^2 = p^2 + \begin{array}{r|r|r|r|l} pq & x + pr & x^2 + ps & x^3 + pt & x^4 +, \text{ \&c.} \\ pq & + q^2 & + qr & + qs & \\ & + pr & + qr & + r^2 & \\ & & + ps & + qs & \\ & & & pt & \end{array}$$

Equating the proper coëfficients, we have

$$\begin{array}{lll} p^2 = 1, & \text{whence} & p = 1. \\ 2pq = 0, & \text{"} & q = 0. \\ 2pr + q^2 = -1, & \text{"} & r = -\frac{1}{2}. \\ 2ps + 2qr = 0, & \text{"} & s = 0. \\ 2pt + 2qs + r^2 = 0, & \text{"} & t = -\frac{1}{8}. \end{array}$$

Therefore, $\sqrt{1 - x^2} = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} -$, &c.

(*Vide* **165**, ex. 9.)

6. Develope $\sqrt{1 - x}$ into a series.

*Ans.* $1 - \frac{x}{2} - \frac{x^2}{2 . 4} - \frac{3x^3}{2 . 4 . 6} - \frac{3 . 5x^4}{2 . 4 . 6 . 8} -$, &c.

7. Develope $\sqrt{1 + x}$ into a series.

*Ans.* $1 + \frac{x}{2} - \frac{x^2}{2 . 4} + \frac{3x^3}{2 . 4 . 6} - \frac{3 . 5x^4}{2 . 4 . 6 . 8}$.

8. If $x = 1$ in the last example, to what does the answer reduce? *Ans.* $\sqrt{2} = 1.41421$.

**239.** 1. Develope the expression $(a + b)^m$ into a series.

*Operation.*

We have $(a + b)^m = a^m (1 + \frac{b}{a})^m$.

For convenience make $x = \frac{b}{a}$, and the expression becomes

$$a^m (1 + x)^m.$$

If now we develope $(1 + x)^m$, and then restore the value of $x$ and multiply by $a^m$, we shall have the development of $(a + b)^m$.

Let $(1 + x)^m = p + qx + rx^2 + sx^3 + tx^4$, &c. (1)

If now $x = 0$, this equation becomes

$$1^m = p; \text{ that is, } p = 1;$$

whence $(1 + x)^m = 1 + qx + rx^2 + sx^3 + tx^4$. (2)

Since (2) is identical, we may have $x = y$;

$$\text{whence} \quad (1+y)^m = 1 + qy + ry^2 + sy^3 + ty^4. \qquad (3)$$

Subtract (3) from (2), and divide by $x - y$, and we have

$$\frac{(1+x)^m - (1+y)^m}{(1+x) - (1+y)} = \frac{q\,(x-y)}{x-y} + \frac{r\,(x^2-y^2)}{x-y} + \frac{s\,(x^3-y^3)}{x-y} + \frac{t\,(x^4-y^4)}{x-y}. \qquad (4)$$

Let now $x = y$, whence $1 + x = 1 + y$, and we have

$$m\,(1+x)^{m-1} = q + 2rx + 3sx^2 + 4tx^3, \text{ \&c.} \qquad (5)$$

Multiply both sides of (5) by $1 + x$, and we have

$$m\,(1+x)^m = \left.\begin{array}{r} q + 2r \\ +q \end{array}\right| x \left.\begin{array}{r} + 3s \\ + 2r \end{array}\right| x^2 \left.\begin{array}{r} + 4t \\ + 3s \end{array}\right| x^3, \text{ \&c.} \qquad (6)$$

Multiply both sides of (2) by $m$, and we have

$$m\,(1+x)^m = m + mqx + mrx^2 + msx^3 + mtx^4, \text{ \&c.} \qquad (7)$$

$$\text{Hence} \quad m + mqx + mrx^2 + msx^3 + \ldots = \left.\begin{array}{r} q + 2r \\ q \end{array}\right| x \left.\begin{array}{r} + 3s \\ 2r \end{array}\right| x^2 \left.\begin{array}{r} + 4t \\ 3s \end{array}\right| x^3 \ldots \qquad (8)$$

Equate the coëfficients of the like powers of $x$, and we have

$m = q$, whence $q = m$.

$mq = 2r + q$, " $r = \dfrac{m\,(m-1)}{1\,.\,2}$.

$mr = 3s + 2r$, " $s = \dfrac{m\,(m-1)\,(m-2)}{1\,.\,2\,.\,3}$.

$ms = 4t + 3s$, " $t = \dfrac{m\,(m-1)\,(m-2)\,(m-3)}{1\,.\,2\,.\,3\,.\,4}$.

Substitute these values in (2), and we have

$$(1+x)^m = 1 + mx + \frac{m\,(m-1)}{1\,.\,2}x^2 + \frac{m(m-1)\,(m-2)}{1\,.\,2\,.\,3}x^3 + \frac{m(m-1)(m-2)(m-3)}{1\,.\,2\,.\,3\,.\,4}x^4, \text{ \&c.} \qquad (9)$$

Substitute for $x$ its equal $\dfrac{b}{a}$, and reduce, and we have

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m\,(m-1)}{1\,.\,2}a^{m-2}b^2 + \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}a^{m-3}b^3, \text{ \&c.} \qquad (10)$$

after multiplying both sides by $a^m$.

This last equation is the *Binomial Theorem,* and it is true for any value of $m$ whatever.

2. Develope $(1+x)^{\frac{1}{2}}$ into a series. Here $m = \frac{1}{2}$. Substitute 1 for $a$, and $x$ for $b$, in equation (10).

$$Ans.\ (1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{2\,.\,4} + \frac{3x^3}{2\,.\,4\,.\,6} - \frac{3\,.\,5x^4}{2\,.\,4\,.\,6\,.\,8}, \text{ \&c.}$$

3. Develope $(1+x)^{-1} = \dfrac{1}{1+x}$ into a series. Here $m = -1$.

$$Ans.\ (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5, \text{ \&c.}$$

4. Develope $(1+x)^{-2} = \dfrac{1}{(1+x)^2}$ into a series. Here $m = -2$.

$$Ans.\ (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4, \text{ \&c.}$$

5. Develope $(1-x)^{-3} = \dfrac{1}{(1-x)^3}$ into a series. Here $m = -3$.

$$Ans.\ (1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4, \text{ \&c.}$$

6. Develope $(1+x)^{\frac{3}{2}}$ into a series. Here $m = \frac{2}{3}$.

$$Ans.\ (1+x)^{\frac{3}{2}} = 1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} -, \text{ \&c.}$$

7. Develope $(a+x)^{\frac{1}{3}}$ into a series.

$$Ans.\ (a+x)^{\frac{1}{3}} = a^{\frac{1}{3}}\left(1 + \frac{x}{3a} - \frac{2x^2}{3\,.\,6a^2} + \frac{2\,.\,5x^3}{3\,.\,6\,.\,9a^3} -, \text{ \&c.}\right)$$

If $a=1$, and $x=1$, then $\sqrt[3]{2} = 1 + \frac{1}{3} - \dfrac{2}{3.6} + \dfrac{2\,.\,5}{3\,.\,6\,.\,9} - \dfrac{2\,.\,5\,.\,8}{3.6.9.12}$, &c.

If $a=1$, and $x=2$, then $\sqrt[3]{3} = 1 + \frac{2}{3} - \dfrac{2.4}{3.6} + \dfrac{2\,.\,5\,.\,8}{3\,.\,6\,.\,9} - \dfrac{2.5.8.16}{3.6.9.12}$, &c.

If $a=8$, and $x=1$, then $\sqrt[3]{9} = 2\left(1 + \frac{1}{24} - \dfrac{2}{3.6.64} + \dfrac{2\,.\,5}{3.6.9.512}, \text{ \&c.}\right)$

8. Develope $(1+x)^{-\frac{1}{2}}$ into a series.

$$Ans.\ (1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} +, \text{ \&c.}$$

## THE TABLE OF LOGARITHMS.

**240.** We propose to show in the present chapter how the table of logarithms has been constructed. For this purpose it

will be sufficient if we actually calculate the logarithms of a few of the lower prime numbers.

In the equation $a^x = N$ (1)

$a$ is the base of the system, and $x$ is the logarithm of the number $N$.

$$\begin{array}{lrcll} \text{Assume} & a &=& 1+m, & \\ \text{and} & N &=& 1+n; & \\ \text{Then} & (1+m)^x &=& 1+n, & (2) \\ \text{and} & (1+m)^{xy} &=& (1+n)^y & (3) \end{array}$$

By the binomial theorem,

$$(1+m)^{xy} = 1 + xym + \frac{xy(xy-1)}{1.2} \cdot m^2 + \frac{xy(xy-1)(xy-2)}{1.2.3} \cdot m^3 +, \&c. \quad (4)$$

$$(1+n)^y = 1 + yn + \frac{y(y-1)}{1.2} n^2 + \frac{y(y-1)(y-2)}{1.2.3} n^3 +, \&c. \quad (5)$$

Equate the right-hand members of (4) and (5), reject the unity, and divide by $y$, and we have

$$x\left(m + \frac{xy-1}{2} \cdot m^2 + \frac{(xy-1)(xy-2)}{2.3} \cdot m^3 +, \&c. = n + \frac{y-1}{2} \cdot n^2 + \frac{(y-1)(y-2)}{2.3} n^3 + \&c.\right) \quad (6)$$

If now $y = 0$, we have

$$x = \log N = \log(1+n) = \frac{n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 +, \&c.}{m - \frac{1}{2}m^2 + \frac{1}{3}m^3 - \frac{1}{4}m^4 +, \&c.} \quad (7)$$

Since $m = a - 1$, and $n = N - 1$, we have

$$\log N = \frac{(N-1) - \frac{1}{2}(N-1)^2 + \frac{1}{3}(N-1)^3 - \frac{1}{4}(N-1)^4 +, \&c.}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 +, \&c.} \quad (8)$$

This equation contains the logarithm of $N$ in terms of $N$ and the base; but, for actual computation, it is necessary to modify its form.

Let the reciprocal of the denominator of (8) be represented by $M$, and replace $n$ for $N - 1$, and we have

$$\log(1+n) = M(n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 +, \&c.) \quad (9)$$

The factor $M$ is called the *modulus* of the system. If $n$ were negative, we should have

$$\log(1-n) = M\left(-n - \tfrac{1}{2}n^2 - \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 -, \text{\&c.}\right) \qquad (10)$$

Subtract (10) from (9), and we have

(*Vide* **138** .) $$\log\frac{1+n}{1-n} = 2M\left(n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \tfrac{1}{7}n^5 +, \text{\&c.}\right) \qquad (11)$$

We may now assume $n = \dfrac{1}{2p+1}$, whence $\dfrac{1+n}{1-n} = \dfrac{p+1}{p}$.

Then

$$\log(p+1) = \log p + 2M\left(\frac{1}{(2p+1)} + \frac{1}{3(2p+1)^3} + \frac{1}{5(2p+1)^5} + \frac{1}{7(2p+1)^7} +, \text{\&c.}\right) \qquad (12)$$

In (12) make $M = 1$, and $p = 1$. Then, since $\log p = \log 1 = 0$, we have

$$\log 2 = 2\left(\tfrac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \frac{1}{9 \cdot 3^9} +, \text{\&c.}\right)$$

The method of summing this is as follows:—

| | | | | |
|---|---|---|---|---|
| 3 | 2. | | | |
| $3^2 = 9$ | 0.66666666 | ÷ 1 | = | 0.66666666. |
| 9 | 0.07407407 | ÷ 3 | = | 0.02469136. |
| 9 | 0.00823045 | ÷ 5 | = | 0.00164609. |
| 9 | 0.00091449 | ÷ 7 | = | 0.00013064. |
| 9 | 0.00010161 | ÷ 9 | = | 0.00001129. |
| 9 | 0.00001129 | ÷ 11 | = | 0.00000103. |
| 9 | 0.00000125 | ÷ 13 | = | 0.00000010. |
| 9 | 0.00000014 | ÷ 15 | = | 0.00000001. |
| | | log 2 | = | 0.69314718. |

In (12) make $M = 1$ and $p = 2$, and we shall find

$\log 3 = 1.098612.$

$2 \log 2 = \log 4 = 1.38629436.$

If $p = 4$, then, $\log 5 = 1.60943790.$

$\text{Log } 5 + \log 2 = \log 10 = 2.30258508.$

Logarithms calculated as above are known as Napierian Logarithms, in honor of Lord Napier, their inventor. It is usual to distinguish them by the contraction *Nap.* Thus,

$$\text{Nap. log } 10 = 2.30258508.$$

We are now to show how common logarithms may be calculated from them.

**241.** Since Napierian logarithms assume the *modulus* to be 1, it follows that if *the Nap. log. of any number is multiplied by the modulus of any other system, the result will be the log. of the same number in that system.*

**242.** To find the *modulus* of the common system of logarithms.

$$\text{Since } \log(1+n) = M\left(n - \frac{n^2}{2} + \frac{n^3}{3} - \frac{n^4}{4}, \text{ \&c.}\right)$$

$$\text{and} \quad \text{Nap. log}(1+n) = n - \frac{n^2}{2} + \frac{n^3}{3} - \frac{n^4}{4}, \text{ \&c.}$$

$$\text{we have} \qquad M = \frac{\log(1+n)}{\text{Nap. log}(1+n)}.$$

That is, the *modulus of any system* is the log. of any number in that system, divided by the Nap. log. of the *same number*.

The log. of the base of any system is 1. Therefore, the base of the common system being 10, we have

$$M = \frac{\log 10}{\text{Nap. log } 10} = \frac{1}{2.30258508}.$$

Hence, the *modulus* of the common system is, 0.4342944819.

**243.** If now the Nap. logs. of 2, 3, 4, 5, &c. are multiplied by 0.4342944819, we shall have

Common log. 2 = 0.301030.
" " 3 = 0.477121.
" " 4 = 0.602060 = 2 log 2.
" " 5 = 0.698970.
&c. &c.

In this way the whole table may be constructed. In practice, the modulus may be retained in the formula to save the trouble of multiplication; and the decimals may be carried to any desirable extent. (*Vide* **153.**)

**244.** To find the Napierian base.

From 242 we must have the following property.

The logs. of the same number in *different systems* are to each other as the *moduli* of the respective systems.

If $a$ represent the base of the Napierian system, then, since its log. is 1, and the modulus of the system 1, we have

$$\text{com. log } a : 1 :: 0.4342944819 : 1.$$

Multiply extremes and means, and we have

$$\text{com. log } a = 0.4342944819.$$

The number in the tables corresponding to this log. is

$$2.718281828459,$$

which is the base of the Napierian system.

**245.** To construct a table of logarithms according to any system whatever.

In the equation $a^x = N$, assume $a$ equal to the desired base, and $N$ equal to the consecutive numbers, and resolve the resulting equations. Thus,

If we desire the base to be 2, make

$$2^x = 1,\ 2^x = 2,\ 2^x = 3,\ 2^x = 4,\ 2^x = 5, \text{ \&c.}$$

and resolve the equations. Thus,

1. $x \log 2 = \log 1$, whence $x = \dfrac{\log 1}{\log 2} = \dfrac{0}{301030} = 0.000000.$

2. $x \log 2 = \log 2$, " $x = \dfrac{\log 2}{\log 2} = \dfrac{301030}{301030} = 1.000000.$

3. $x \log 2 = \log 3$, whence $x = \frac{\log 3}{\log 2} = \frac{477121}{301030} = 1.584558$

&c. &c.

By continuing the process, we should form a table of logs. with 2 as a base.

It is evident that the base cannot be a *negative number*.

## PRACTICAL APPLICATIONS.

**246.** 1. Solve the equation $7^x = 13$.

$$\textit{Ans. } x = \frac{\log 13}{\log 7} = 1.3181.$$

2. Solve the equation $\left(\frac{3}{5}\right)^x = \frac{2}{3}$.

$$\textit{Ans. } x = \frac{\log 2 - \log 3}{\log 3 - \log 5} = .7937.$$

3. Solve the equation $mn^x = a$.

$$\textit{Ans. } x = \frac{\log a - \log m}{\log n}.$$

4. Find the value of $n$ in the equation $l = ar^{n-1}$.

$$\log l = \log a + (n - 1) \log r.$$

whence $$n - 1 = \frac{\log l - \log a}{\log r}.$$

Therefore, $$n = \frac{\log l - \log a}{\log r} + 1.$$ (*Vide* **231**, form. 17.)

5. Find the value of $n$ in the equation $s = \frac{a(r^n - 1)}{r - 1}$.

We have $s(r - 1) = ar^n - a$,

or, $a + (r - 1)s = ar^n$;

whence $\log [a + (r - 1)s] = \log a + n \log r$.

Therefore, $$n = \frac{\log [a + (r - 1)s] - \log a}{\log r}.$$

(*Vide* **231**, form. 18.)

6. Find the value of $n$ in the equation $a\,(s - a)^{n-1} = l\,(s - l)^{n-1}$.

We have $\log a + (n - 1) \log (s - 1) = \log l + (n - 1) \log (s - l)$.

Therefore, $$n = \frac{\log l - \log a}{\log (s - a) - \log (s - l)} + 1.$$

Vide **231**, form. 19.)

7. Find $n$ in the equation $(s - l)\, r^n - s r^{n-1} + l = 0$.

This is the same as $r\,(s - l) r^{n-1} - s r^{n-1} + l = 0$,

which is $[rl - (r - 1)\, s]\, r^{n-1} = l$.

The log of this is $\log\, [rl - (r - 1)\, s] + (n + 1) \log r = \log l$.

whence $$n = \frac{\log l - \log\, [rl - (r - 1)\, s]}{\log r} + 1$$

Vide **231**, form. 20.)

# CHAPTER VIII.

## EQUATIONS OF THE THIRD DEGREE.

**247.** The general form of equations of this degree is

$$x^3 + px^2 + qx = m. \qquad (1)$$

In which $p$, $q$, and $m$ may be positive or negative.

**248.** If $p = 0$, $q = 0$, and $m = a^3$, we have

$$x^3 = a^3, \text{ or } x^3 - a^3 = 0.$$

By 78, this equation may be written thus:

$$(x - a)(x^2 + ax + a^2) = 0.$$

Divide by each factor of the first member, and we have

$$x - a = 0 \text{ and } x^2 + ax + a^2 = 0.$$

From the first of these we have

$x = a$; and from the second, $x = \frac{a}{2}(-1 \pm \sqrt{-3})$. (*Vide* 167.)

Hence the equation has three roots, two of which are imaginary.

**249.** If $p = 0$, $q = 0$, and $m = -a^3$, we have

$$x^3 = -a^3, \text{ or } x^3 + a^3 = 0.$$

By 79, this may be written thus:

$$(x + a)(x^2 - ax + a^2) = 0.$$

Hence $x = -a$, and $x = \frac{a}{2}(1 \pm \sqrt{-3})$. (*Vide* 167.)

**250.** 1. Given $x^3 = 1$, to find the values of $x$.

Here $\qquad x^3 - 1 = 0;$

Hence $\qquad (x-1)(x^2 + x + 1) = 0;$

Whence $\qquad x = 1$ and $x = \frac{1}{2}(-1 \pm \sqrt{-3})$.

2. Given $x^3 = 8$, to find the values of $x$.

*Ans.* $x = 2$ and $-1 \pm \sqrt{-3}$.

3. Given $x^3 = 27$, $x^3 = 64$, $x^3 = -125$, and $x^3 = -216$, to find, &c.

4. Given $x^3 = 10$, to find the values of $x$.

*Ans.* $x = \sqrt[3]{10}$ and $\sqrt[3]{\frac{10}{8}}(-1 \pm \sqrt{-3})$,
or $x = 2.1544$ and $-1.0772 \pm 1.8657\sqrt{-1}$.

**251.** IF, IN THE EQUATION $x^3 + px^2 + qx + m = 0$, $a$ IS AN EXACT ROOT, THEN THE FIRST MEMBER IS EXACTLY DIVISIBLE BY $x - a$.

For, since $a$ is an exact root, we may substitute it for $x$ in the given equation, and write

$$a^3 + pa^2 + qa + m = 0. \qquad (2)$$

Now let us proceed to divide the given equation by $x - a$.

$$\begin{array}{l|l}
x^3 + px^2 + qx + m & x - a \\ \hline
x^3 - ax^2 & x^2 + (p+a)x + q + a(p+a) \\ \hline
\quad (p+a)x^2 + qx & \\
\quad \underline{(p+a)x^2 - a(p+a)x} & \\
\qquad\qquad (q + a(p+a))x + m & \\
\qquad\qquad \underline{(q + a(p+a))x - aq - a^2(p+a)} &
\end{array}$$

The last remainder is $aq + a^2(p+a) + m = a^3 + pa^2 + qa + m$.

But, by (2), this remainder is 0. Therefore the quotient is exact.

**252.** It is evident that the same reasoning applies to any equation of the form

$$x^n + px^{n-1} + qx^{n-2} \text{ \&c.} + tx + m = 0. \qquad (3)$$

For, since the dividend is equal to the divisor multiplied by the quotient plus the remainder, if we denote the quotient by $l$, and a supposed remainder, on division, by $r$, we should have

$$x^n + px^{n-1} + qx^{n-2} \text{ \&c.} + tx + m = (x - a)\, l + r. \qquad (4)$$

Now, if $a$ is substituted for $x$, the first member of this equation is 0. But the first term of the second member is 0, and $\therefore$ $r = 0$.

HENCE, IF $a$ IS A ROOT OF ANY EQUATION OF THE FORM OF (3), THAT EQUATION IS DIVISIBLE BY $x - a$.

EXAMPLES.

**253.** 1. One root of the equation $x^3 - 6x^2 + 11x = 6$ is 1: what are the other roots?

We may write $x^3 - 6x^2 + 11x - 6 = 0$, which, by 251, is exactly divisible by $x - 1$. On dividing, the factors of the equation are found to be $(x - 1)(x^2 - 5x + 6) = 0$.

Hence $\qquad x^2 - 5x = -6;$

From which $\qquad x = 3$, or $x = 2$.

2. One root of $x^3 + 4x^2 - 7x = 190$ is 5: what are the other roots? *Ans.* $\frac{1}{2}(-9 \pm \sqrt{-71})$.

3. One root of $x^3 - 4x^2 - 11x = -30$ is 2: what are the other roots? *Ans.* $x = 5$ and $x = -3$.

4. One root of $x^3 + x^2 - 22x = 40$ is 5: what are the other roots? *Ans.* $x = -4$ and $x = -2$.

5. One root of $x^3 + 2x^2 = 16$ is 2: what are the other roots? *Ans.* $2(-1 \pm \sqrt{-1})$.

6. One root of $x^3 - 2x^2 - 2x = -4$ is 2: what are the other roots? *Ans.* $\pm 1.41421$.

7. One root of $x^3 - 2x = -\frac{7}{8}$ is $\frac{1}{2}$: what are the other roots? *Ans.* 1.0962 and $-1.5962$.

8. One root of $x^3 - 6x^2 + x = -28$ is 4: what are the other roots? *Ans.* 3.82841 and $-1.82841$.

9. One root of $x^3 + 9x^2 + 26x = -24$ is $-4$: what are the other roots? *Ans.* $-2$ and $-3$.

**254.** IF THE EQUATION $x^3 + px^2 + qx + m = 0$ IS DIVISIBLE BY $x - a$, THEN $a$ IS A ROOT OF THE EQUATION.

For, in this case, the remainder is 0, and, by § 251, we may write

$$x^3 + px^2 + qx + m = (x - a)\,(x^2 + (p + a)x + q + a\,(p + a)\,).$$

If $x = a$, the second member reduces to 0; $\therefore$ $a$ substituted for $x$ will reduce the first member to 0. Hence $a$ is a root of the given equation.

**255.** THE SAME PROPOSITION IS TRUE OF THE EQUATION $x^n + px^{n-1} + qx^{n-2}$ &c. $+ tx + m = 0$.

For in this case we may write, by § 252,

$$x^n + px^{n-1} + qx^{n-2} \text{ \&c. } + tx + m = (x - a)l.$$

If $x = a$, the second member reduces to 0; $\therefore$ $a$ substituted for $x$ will reduce the first member to 0. Hence $a$ is a root of the given equation.

**256.** Hence, to ascertain if a polynomial containing $x$ is exactly divisible by $x - a$, *substitute a for x, and if the polynomial reduces to* 0, *the division is exact.*

EXAMPLES.

1. The equation $x^3 - 12x^2 + 47x - 60 = 0$ has three roots less than 10: what are they? *Ans.* 3, 4, and 5.

2. The equation $x^3 + 9x^2 + 26x + 24 = 0$ has three negative roots less than $-10$: what are they? *Ans.* $-2$, $-3$, and $-4$.

3. The equation $x^3 - 6x^2 + 11x - 6 = 0$ has three roots less than 10: what are they? *Ans.* 1, 2, and 3.

4. The equation $x^3 + 13x^2 + 44x + 32 = 0$ has three negative roots less than $-10$: what are they?

*Ans.* $-1$, $-4$, and $-8$.

**257.** THE EQUATION $x^3 + px^2 + qx + m = 0$ HAS THREE ROOTS, AND ONLY THREE.

For, if $a$ is one root of the equation, its factors, by § 254, are

$$(x - a)\,(x^2 + (p + a)x + q + a\,(p + a)) = 0.$$

Dividing first by one factor and then by the other, we have

$$x - a = 0, \text{ and } x^2 + (p + a)x + q + a\,(p + a) = 0.$$

The roots of which are $x = a$, and

$$x = -\tfrac{1}{2}\,((p + a) \pm \sqrt{(p - a)^2 - 4\,(a^2 + q)}). \qquad (5)$$

**258.** THE EQUATION $x^n + px^{n-1} + qx^{n-2}$ &c. $+ tx + m$ HAS $n$ ROOTS, AND ONLY $n$.

For, if the equation were of the 4th degree, *i.e.* $n = 4$, on dividing by $x - a$, $a$ being one root, it would be reduced to an equation of the 3d degree, which, by § 257, has 3 roots; ∴ the equation itself has 4 roots.

Since the same reasoning applies to an equation of the 5th, 6th, 7th, &c. degree, we conclude generally that AN EQUATION OF THE $n$TH DEGREE HAS $n$ ROOTS.

EXAMPLES.

1. One root of $x^3 - 11x^2 + 16x + 84 = 0$ is $-2$: what are the other roots?

Here $a = -2$, $p = -11$, and $q = 16$, and formula (5) gives $x = 6$, and $x = 7$.

2. One root of $x^3 + 7x^2 - 4x - 28 = 0$ is $-7$: what are the other roots? *Ans.* $x = \pm 2$.

3. One root of $x^3 + 13x^2 + 44x + 32 = 0$ is $-1$: what are the other roots? *Ans.* $-4$ and $-8$.

4. One root of $x^4 - 12x^3 + 28x^2 + 68x - 84 = 0$ is 1: what are the other roots? *Ans.* ———.

5. One root of $x^4 + 4x^3 - 25x^2 - 16x + 84 = 0$ is 3: what are the other roots? *Ans.* ———.

6. One root of $x^5 + 9x^4 - 5x^3 - 141x^2 + 4x + 420 = 0$ is $-5$: what are the other roots? *Ans.* ———.

**259.** If $a$, $b$, and $c$ are the roots of $x^3 + px^2 + qx + m = 0$, then we shall have

$$x^3 + px^2 + qx + m = (x - a)(x - b)(x - c). \qquad (6)$$

For, since the given equation is divisible by either of the expressions $(x - a)$, $(x - b)$, or $(x - c)$, it must be composed of these three factors, and none others, considering, also, that it can have only three roots.

**260.** In the same manner, if $a$, $b$, $c$, .... $k$, and $l$ are the roots of the equation $x^n + px^{n-1} + qx^{n-2}$ &c. $+ tx + m$, then we shall have

$$x^n + px^{n-1} + qx^{n-2} \text{ \&c. } + tx + m = (x - a)(x - b)(x - c) \ldots\ldots (x - k)(x - l). \qquad (7)$$

EXAMPLES.

1. Find the equation whose roots are 1, 2, and 3.
*Ans.* $(x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6 = 0$.

2. Find the equation whose roots are 1, 2, and $-5$.
*Ans.* $x^3 + 2x^3 - 13x + 10 = 0$.

3. Find the equation whose roots are $\sqrt{2}$, $-\sqrt{2}$, and 2.
*Ans.* $x^3 - 2x^2 - 2x = -4$.

4. Find the equation whose roots are $-6$, $-\sqrt{3}$, and $\sqrt{3}$.

*Ans.* $x^3 + 6x^2 - 3x = 18$.

5. Find the equation whose roots are 3, $1 + \sqrt{3}$, and $1 - \sqrt{3}$.

*Ans.* $x^3 - 5x^2 + 4x = -6$.

**261.** If the second member of (6) is developed by actual multiplication, we have $x^3 + px^2 + qx + m = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$. (8)

Whence $p = -(a + b + c)$, $q = ab + ac + bc$, and $m = -abc$.

(*Vide* 237.)

That is,—

1. The coefficient of the first term is 1.

2. The coëfficient of the second term is the algebraic sum of the roots with a contrary sign.

3. The coëfficient of the third term is the algebraic sum of the roots taken in products, as many times as there are different sets of *two roots* in each set.

4. The term independent of $x$ is the product of the roots with a contrary sign.

EXAMPLES.

1. Find the equation whose roots are 1, 2, and 3.

Here $p = -(1 + 2 + 3)$, $q = 1 \times 2 + 1 \times 3 + 2 \times 3$

and $m = -1 \times 2 \times 3$.

Hence the equation is $x^3 - 6x^2 + 11x - 6 = 0$.

(*Vide* § **260**, Ex. 1.)

2. Find the equation whose roots are 3, $3 + \sqrt{3}$, and $3 - \sqrt{3}$.

Here $p = -9$, $q = 24$, and $m = -18$.

Hence the equation is $x^3 - 9x^2 + 24x - 18 = 0$.

3. Find the equation whose roots are 1, 2, and $-3$.

Here $p = 0$, $q = -7$, and $m = 6$.

Hence the equation is $x^3 - 7x + 6 = 0$.

4. Find the equation whose roots are $-\frac{1}{2}$, $-\frac{1}{3}$, and $-\frac{1}{4}$.

Here $p = \frac{13}{12}$, $q = \frac{9}{24}$, and $m = \frac{1}{24}$.

Hence the equation is $x^3 + \dfrac{13x^2}{12} + \dfrac{9x}{24} + \dfrac{1}{24} = 0$.

5. Find the equation whose roots are 5, $5 + \sqrt{-1}$, and $5 - \sqrt{-1}$. *Ans.* $x^3 - 15x^2 + 76x - 130 = 0$.

**262.** If the second member of equation (7) were developed by actual multiplication, we should find the law for the coëfficients of the first three terms the same as before, while the coëfficient of the fourth term *will be the algebraic sum of the products of the roots taken as many times as there are different sets with three roots in each set, the sign of the final result being changed.*

The coëfficient of the fifth term *will be the algebraic sum of all the products of the roots, taken as many times as there are sets with four roots in each set.*

And, in general, the coëfficient of the $n$th term *will be the algebraic sum of all the products of the roots, taken as many times as there are sets with $n - 1$ roots in each set,* the sign of the final result being changed when $n$ is *even* and retained when $n$ is *odd.*

This $n$ must not be confounded with the $n$ which denotes the degree of the equation.

EXAMPLES.

1. Find the equation whose roots are $-2$, $-2$, 4, and $-4$.

Here $p = 4$, $q = -12$, $t = -64$, and $m = -64$.

Hence the equation is $x^4 + 4x^3 - 12x^2 - 64x - 64 = 0$.

2. Find the equation whose roots are 3, 4, $-1$, and $-6$.

*Ans.* $x^4 - 31x^2 + 42x + 72 = 0$.

3. Find the equation whose roots are 1, 2, 3, 4, and 5.

*Ans.* $x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 = 0$.

**263.** THE EQUATION $x^3 + px^2 + qx + m = 0$ MAY BE TRANSFORMED INTO ANOTHER OF THE SAME FORM, IN WHICH THE

ROOTS ARE ANY GIVEN MULTIPLE OF THOSE OF THE GIVEN EQUATION.

For, in the equation $x^3 + px^2 + qx + m = 0$, substitute $\frac{y}{k}$ for $x$, and we have $\frac{y^3}{k^3} + \frac{py^2}{k^2} + \frac{qy}{k} + m = 0$, which multiplied by $k^3$ gives

$$y^3 + kpy^2 + k^2qy + k^3m = 0, \qquad (9)$$

which is the equation it was proposed to obtain. The roots of (9) are $k$ times the roots of the given equation; since $y = kx$.

EXAMPLES.

1. Transform the equation $x^3 - 6x^2 + 11x - 6 = 0$ into another whose roots are 3 times as large.

Here $x = \frac{y}{3}$, i.e. $y = 3x$. Make $k$ of (9) $= 3$, and we have $y^3 - 18y^2 + 99y - 162 = 0$, where $p = -6$, $q = 11$, $m = -6$.

2. Transform the equation $x^3 - 6x^2 + 11x - 6 = 0$ into another whose roots are $\frac{1}{2}$ as large. Here $k = \frac{1}{2}$.

*Ans.* $y^3 - 3y^2 + \frac{11y}{4} - \frac{3}{4} = 0$.

**264.** IN THE EQUATION $x^3 + px^2 + qx + mx^0 = 0$, IF THE COËFFICIENTS ARE FRACTIONAL, THE EQUATION MAY BE TRANSFORMED INTO ONE WHOSE COËFFICIENTS ARE ENTIRE.

For, in equation (9) it is manifest that $k$ may be so taken that all the coëfficients may be entire if either $p$, $q$, or $m$ are fractions.

EXAMPLES.

1. Transform the equation $x^3 + \frac{7x^2}{8} + \frac{7x}{32} + \frac{1}{64} = 0$ into one whose coëfficients are entire. Here $p = \frac{7}{8}$, $q = \frac{7}{32}$, and $m = \frac{1}{64}$. Make $k = 8$, and equation (9) becomes $y^3 + 7y^2 + 14y + 8 = 0$, the roots of which are 8 times those of the given equation.

2. Transform $x^3 + \frac{x^2}{4} + \frac{x}{16} + \frac{1}{64} = 0$ into an equation whose coëfficients are entire. Make $k = 4$.

*Ans.* $y^3 + y^2 + y + 1 = 0.$

3. Transform $x^3 - 3x^2 + \frac{11x}{4} - \frac{3}{4} = 0$ into one whose coëfficients are entire. Make $k = 2$.

*Ans.* $y^3 - 6y^2 + 11y - 6 = 0.$

4. Transform $x^3 - \frac{2x^2}{25} + \frac{x}{30} - \frac{1}{40} = 0$ into one whose coëfficients are entire. Make $k = 150$.

*Ans.* $y^3 - 12y^2 + 750y - 84375 = 0.$

**265.** THE PROPOSITIONS OF §§ 263 AND 264 ARE EQUALLY APPLICABLE TO THE EQUATION $x^n + px^{n-1} + qx^{n-2} \ldots + tx + m = 0$.

For, make $x = \frac{y}{k}$, and we have

$\frac{y^n}{k^n} + p\frac{y^{n-1}}{k^{n-1}} + q\frac{y^{n-2}}{k^{n-2}} \ldots + t\frac{y}{k} + m = 0.$ Multiply this by $k^n$, and we have

$$y^n + pky^{n-1} + qk^2y^{n-2} \ldots + tk^{n-1} + mk^n = 0. \qquad (9)$$

In this equation, $k$ may always be so taken that the fractions will all disappear.

EXAMPLES.

1. Transform $x^4 - \frac{5x^3}{6} + \frac{5x^2}{12} - \frac{7x}{150} - \frac{13}{9000}$ into one whose coëfficients are entire.

Make $k = 30$, and we have

$$y^4 - 25y^3 + 375y^2 - 1260y - 1170 = 0.$$

The roots of this equation are 30 times those of the given equation.

2. Transform $x^5 - \frac{13x^4}{12} + \frac{21x^3}{40} - \frac{32x^2}{225} - \frac{43x}{600} - \frac{1}{800} = 0.$

Make $k = 60$, and we have

$$y^5 - 65y^4 + 1890y^3 - 30720y^2 - 928800y + 972000 = 0.$$

**266.** THE EQUATION $x^3 + px^2 + qx + m = 0$ MAY BE TRANSFORMED INTO ANOTHER EQUATION OF THE SAME FORM, WHOSE ROOTS *differ* FROM THOSE OF THE GIVEN EQUATION BY A GIVEN QUANTITY.

For, let $x = y + r$. Substitute this value of $x$ in the given equation, and we have

$(y + r)^3 + p(y + r)^2 + q(y + r) + m = 0$. By expanding the terms, we have

$$y^3 + 3y^2r + 3yr^2 + r^3 + py^2 + 2pry + pr^2 + qy + qr + m = 0.$$

Arrange the terms according to the powers of $y$, and we have

$$y^3 + (3r + p)y^2 + (3r^2 + 2pr + q)y + r^3 + pr^2 + qr + m = 0 \quad (10).$$

Which is the equation it was proposed to obtain, since $y = x - r$.

If we write the coëfficients of this equation above each other, commencing with the last, we have

$$\begin{array}{ll} r^3 + pr^2 + qr + m, & (D_0). \\ 3r^2 + 2pr + q, & (D_1). \\ 3r + p, & (D_2). \end{array}$$

If the term independent of $r$ be considered as the coëfficient of $r^0$, these coëfficients are *derived* in the following way:—

The *first coëfficient* is what the given equation becomes when $r$ is substituted for $x$.

The *second coficient* is *derived* from the first by multiplying the coëfficient of each term by the exponent of $r$ in that term, and then diminishing the exponent by 1. It is called the *First Derived Polynomial.*

The *third coëfficient* is *derived* from the second in the same way, except that each product is divided by 2. It is called the *Second Derived Polynomial.*

## EXAMPLES.

1. Transform the equation $x^3 - 6x^2 + 11x - 6 = 0$ into one whose roots are less than those of the given equation by 4.

Here $r^3 + pr^2 + qr + m = (4)^3 - 6(4)^2 + 11(4) - 6 = 6.$

$$3r^2 + 2pr + q = 3(4)^2 - 12(4) + 11 = 11.$$

$$3r + p = \tfrac{6}{2}(4) - \tfrac{12}{2} = 6.$$

Hence the equation is $y^3 + 6y^2 + 11y + 6 = 0$.

The roots of the given equation are 1, 2, and 3. Those of the transformed equation are $-3$, $-2$, and $-1$.

2. Transform the equation $x^3 - 12x^2 + 47x = 60$ into one whose roots are less than those of the given equation by 1.

*Ans.* $y^3 - 9y^2 + 26y = 24$.

3. Transform $x^3 - 9x^2 + 26x = 24$ into one whose roots are less than those of the given equation by 1.

*Ans.* $y^3 - 6y^2 + 11y = 6$.

4. Transform $x^3 - 6x^2 + 11x = 6$ into one equation whose roots are less than those in the given equation by 1.

*Ans.* $y^3 - 3y^2 + 2y = 0$.

5. Transform $x^3 + 13x^2 + 44x + 32 = 0$ into an equation whose roots are greater than those in the given equation by 10.

(*Vide* **256**, Ex. 4.) *Ans.* $y^3 - 17y^2 + 84y = 108$.

**267.** The proposition of § 266 is equally applicable to the equation $x^n + px^{n-1} + qx^{n-2} \ldots + tx + m = 0$. For, make $x = y + r$, substitute in the given equation, develope the several terms by the binomial theorem, and arrange the terms according to the powers of $y$, and the coëfficients will become as follows:—

Of $y^0$ it is $r^n + pr^{n-1} + qr^{n-2} \ldots . tr + m$ $(D_0)$.

Of $y^1$ it is $nr^{n-1} + (n-1)pr^{n-2} + (n-2)qr^{n-3} \ldots . + t$ $(D_1)$.

Of $y^2$ it is

$$\frac{n(n-1)}{1.2}r^{n-2} + \frac{(n-1)(n-2)}{1.2}r^{n-3} + \frac{(n-2)(n-3)}{1.2}r^{n-4} \ldots (D_2).$$

Of $y^3$ it is

$$\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} r^{n-3} + \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} r^{n-4}, \&c. \quad (D_3).$$

Of $y^4$ it is $\dfrac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} r^{n-4}$, &c. $(D_4)$.

Each of these coëfficients is derived from the one immediately preceding, according to the following *law of derived polynomials:*—

MULTIPLY EACH TERM IN SUCCESSION BY THE EXPONENT OF $r$ IN THAT TERM, DIVIDE THE PRODUCT BY THE NUMBER WHICH DESIGNATES THE PLACE OF THE COËFFICIENT, AND DIMINISH THE EXPONENT OF $r$ BY UNITY.

EXAMPLES.

1. Transform the equation $3x^4 - 4x^3 + 7x^2 + 8x - 12 = 0$ into one whose roots are less than those in the given equation by 3.

$$\begin{aligned}
\text{Here } (D_0) &= 3(3)^4 - 4(3)^3 + 7(3)^2 + 8(3) - 12 &&= 210, \\
(D_1) &= 12(3)^3 - 12(3)^2 + 14(3) + 8 &&= 266, \\
(D_2) &= 18(3)^2 - 12(3) + 7 &&= 133, \\
(D_3) &= 12(3) - 4 &&= 32, \\
(D_4) &= 3 &&= 3.
\end{aligned}$$

Hence the equation is, $3y^4 + 32y^3 + 133y^2 + 266y + 210 = 0$.

2. Transform $x^4 - 31x^2 + 42x + 72 = 0$ into one whose roots are greater by 5. *Ans.* $y^4 - 20y^3 + 119y^2 - 148y = 216$.

**268.** THE EQUATION $x^3 + px^2 + qx + m = 0$ MAY BE TRANSFORMED INTO ANOTHER EQUATION WHOSE SECOND OR THIRD TERM IS WANTING.

For, to make the second term disappear, make, in (10),

$$3r + p = 0, \text{ or } r = \frac{-p}{3}.$$

To make the third term disappear, make, in (10),

$$3r^2 + 2pr + q = 0; \text{ i.e. } r = \tfrac{1}{3}(-p \pm \sqrt{p^2 - 3q}).$$

EXAMPLES.

1. Make the second term disappear in the equation $x^3 - 6x^2 + 11x - 6 = 0$.

Here $r = \frac{6}{3} = 2$. Therefore the coëfficients of (10) become

$$\begin{aligned} r^3 + pr^2 + qr + m &= (2)^3 - 6(2)^2 + 11(2) - 6 = 0. \\ 3r^2 + 2pr + q &= 3(2)^2 - 12(2) + 11 = -1. \\ 3r + p &= 3(2) - 6 = 0. \end{aligned}$$

Hence the equation is $y^3 - y = 0$.

The roots of this equation are, of course, less than those of the given equation by 2. The roots of the given equation are 1, 2, and 3. $\therefore$

Those of the transformed equation ought to be $-1$, 0, and $+1$. And we really have $y^3 - y = y(y^2 - 1) = 0$; whence $y = 0$ and $y = \pm 1$.

It may frequently happen that more than one term disappears at a time, as in the example above.

2. Make the *third* term disappear in the equation $x^3 - 6x^2 + 11x - 6 = 0$.

Here $r = 2 \pm \frac{1}{3}\sqrt{3}$. The coëfficients of (10) become

$$\begin{aligned} (2 \pm \tfrac{1}{3}\sqrt{3})^3 - 6(2 \pm \tfrac{1}{3}\sqrt{3})^2 + 11(2 \pm \tfrac{1}{3}\sqrt{3}) - 6 &= \mp \tfrac{2}{9}\sqrt{3}, \\ 3(2 \pm \tfrac{1}{3}\sqrt{3})^2 - 12(2 \pm \tfrac{1}{3}\sqrt{3}) + 11 &= 0, \\ 3(2 \pm \tfrac{1}{3}\sqrt{3}) - 6 &= \pm\sqrt{3}. \end{aligned}$$

Hence the equation is $y^3 \pm \sqrt{3}.\, y^2 \mp \frac{2}{9}\sqrt{3} = 0$.

3. Make the second term disappear in the equation $x^3 - 14x^2 + 61x = 84$.

*Ans.* $y^3 - \dfrac{13y}{3} = -\frac{70}{27}$.

4. Make the second term disappear in the equation $x^3 - 12x^2 + 47x = 60$.

*Ans.* $y^3 - y = 0$.

The roots of the resulting equation are $-[illegible]$, [illegible], and $-[illegible]$. The roots of the given equation must therefore be greater by [illegible]; that is, they are [illegible], [illegible], and [illegible]. (Art. 256, Ex. 1.)

[illegible]. What are the roots of the equation $x^3 - [illegible]x^2 - [illegible]x = [illegible]$?

First make the second term disappear, and we have $y^3 - [illegible]y = [illegible]$, the roots of which are $-[illegible]$, [illegible], and $-[illegible]$.

The roots of the given equation must be greater by [illegible]. They are therefore [illegible], [illegible], and [illegible].

[illegible]. What are the roots of the equation $x^3 - [illegible]x^2 - [illegible]x = [illegible]$?

Ans. [illegible], [illegible], [illegible].

[illegible]. What are the roots of the equation $x^3 - [illegible]x^2 - [illegible]x = -[illegible]$? Here $h = -[illegible]$.

Ans. $-[illegible]$, $-[illegible]$, and $-[illegible]$.

[illegible]. What are the roots of the equation $x^3 - [illegible]x^2 - [illegible]x = -[illegible]$?

Ans. [illegible], $-[illegible]$, and [illegible].

[illegible]. What are the roots of the equation $x^3 - [illegible]x^2 - [illegible]x = [illegible]$?

Ans. [illegible], $-[illegible]$, and [illegible].

## PROPERTIES OF DERIVED POLYNOMIALS.

**269.** By § [illegible], we have

$$x^3 - px^2 - qx - n = (x - a)(x - b)(x - c) \ldots [illegible]$$

Making $x = y - h$, we have, by considering $h - a$, $h - b$, and $h - c$ as single quantities, $(y - h)^3 - p(y - h)^2 - q(y - h) - n = (y - \overline{h - a})(y - \overline{h - b})(y - \overline{h - c})$.

By §§ [illegible] and [illegible], we have the following identical equation:—

[illegible]

In this equation the coefficients of the like powers of $y$ are equal. Hence we have

$$r^3 + pr^2 + qr + m = (r-a)(r-b)(r-c).$$
$$3r^2 + 2pr + q = (r-a)(r-b) + (r-a)(r-c) + (r-b)(r-c).$$
$$3r + p = (r-a) + (r-b) + (r-c).$$
$$1 = 1.$$

Since $r$ is arbitrary, we may make it equal to $x$, and thus obtain

$$x^3 + px^2 + qx + m = (x-a)(x-b)(x-c) \qquad (6),$$
$$(D_1) = 3x^2 + 2px + q = (x-a)(x-b) + (x-a)(x-c) + (x-b)(x-c) \ (d_1),$$
$$(D_2) = \qquad 3x + p = (x-a) + (x-b) + (x-c) \qquad (d_2).$$

Equation $(d_1)$ proves that the first derivative is equal to the sum of the products of the several factors of (6) taken as many times as there are different sets with two factors in each set.

Equation $(d_2)$ proves that the second derivative is equal to the sum of the several factors of (6).

By taking the same steps with the equation

$$x^n + px^{n-1} + qx^{n-2} \ldots + tx + m = (x-a)(x-b)(x-c)\ldots(x-l) \ (12),$$

we should find the following relations:—

THE FIRST DERIVED POLYNOMIAL, THAT IS, $D_1$, IS EQUAL TO THE SUM OF THE PRODUCTS OF THE $n$ FACTORS OF (12) TAKEN AS MANY TIMES AS THERE ARE SETS WITH $n-1$ FACTORS IN EACH SET.

THE SECOND DERIVED POLYNOMIAL, THAT IS, $D_2$, IS EQUAL TO THE SUM OF THE PRODUCTS OF THE $n$ FACTORS OF (12) TAKEN AS MANY TIMES AS THERE ARE SETS WITH $n-2$ FACTORS IN EACH SET.

THE DERIVED POLYNOMIAL, $D_{n-1}$, IS EQUAL TO THE SUM OF THE $n$ FACTORS OF (12).

## EQUAL ROOTS.

**270.** If, in (6) and $(d_1)$, § 269, we make $a = b$, the equations become $x^3 + px^2 + qx + m = (x-a)^2(x-c)$,

and $\qquad 3x^2 + 2px + q = (x-a)^2 + 2(x-a)(x-c)$,

the second members of which are each divisible by $x - a$. Hence,

If the equation $x^3 + px^2 + qx + m = 0$ contains equal roots, the equation and its derived polynomial will have a common divisor containing that root; and, conversely,

If the equation and its derived polynomial have a common divisor, the equation has equal roots.

If we make $a = b = c$, the equations become

$$x^3 + px^2 + qx + m = (x - a)^3,$$

and $$3x^2 + 2px + q = 3(x - a)^2,$$

the second members of which are each divisible by $(x - a)^2$. Hence,

If the equation has *three* equal roots, the greatest common divisor of the equation and its derived polynomial contains *two* of these equal roots.

And, conversely, if the greatest common divisor contains *two* equal roots, the given equation contains *three* roots of the same value. And, generally,

IF THE EQUATION $x^n + px^{n-1} + qx^{n-2} \ldots + tx + m = 0$ CONTAINS EQUAL ROOTS TO THE NUMBER OF $s$, THE EQUATION AND ITS DERIVED POLYNOMIAL HAVE A COMMON DIVISOR CONTAINING $s - 1$ OF THESE EQUAL ROOTS; or,

IF THE GREATEST COMMON DIVISOR OF AN EQUATION AND ITS DERIVED POLYNOMIAL CONTAINS $s - 1$ EQUAL ROOTS, THE EQUATION CONTAINS $s$ ROOTS OF THE SAME VALUE.

## EXAMPLES.

1. Find the roots of the equation $x^3 - 11x^2 + 32x - 28 = 0$.

The FIRST DERIVATIVE is $3x^2 - 22x + 32 = 0$; the roots of which are $x = 2$ and $\frac{16}{3}$.

Of the factors $x - 2$ and $x - \frac{16}{3}$ the former is that which will divide the given equation (*Vide* § 256); $\therefore$ the roots are 2, 2, and 7.

And, in fact, $(x - 2)^2 (x - 7) = x^3 - 11x^2 + 32x - 28$.

2. Find the roots of the equation $x^3 - 10x^2 + 33x - 36$.
(*Vide* 84, Ex. 8.) *Ans.* 3, 3, and 4.

3. Find the roots of the equation $x^3 - 13x^2 + 56x - 80$.
(*Vide* 84, Ex. 9.) *Ans.* 4, 4, and 5.

4. Find the roots of the equation $x^4 - 5x^3 + 9x^2 - 7x + 2$.
(*Vide* 84, Ex. 6.) *Ans.* 1, 1, 1, and 2.

5. Find the roots of the equation $x^4 - 10x^3 + 37x^2 - 60x + 36$.
(*Vide* 84, Ex. 7.) *Ans.* 3, 3, 2, and 2.

## GENERAL SOLUTION OF THE EQUATION OF THE THIRD DEGREE.

**271.** By § 268, any equation of this degree may take the form

$$x^3 + qx + m = 0. \tag{1}$$

If $x = y + r$, we have $x^3 = y^3 + r^3 + 3yr\,(y + r)$;
Or, which is the same thing, $x^3 = y^3 + r^3 + 3yr.x$.

Hence
$$x^3 - 3yr.x - y^3 - r^3 = 0. \tag{2}$$
Therefore
$$x^3 + qx + m = x^3 - 3yr.x - y^3 - r^3. \tag{3}$$
Hence, by § 237, $y^3 + r^3 = -m$, and $3yr = -q$.

Whence $y = \sqrt[3]{-\frac{m}{2} + \sqrt{\frac{m^2}{4} + \frac{q^3}{27}}}$, and $r = \sqrt[3]{-\frac{m}{2} - \sqrt{\frac{m^2}{4} + \frac{q^3}{27}}}$.

Therefore
$$x = \sqrt[3]{-\frac{m}{2} + \sqrt{\frac{m^2}{4} + \frac{q^3}{27}}} + \sqrt[3]{-\frac{m}{2} - \sqrt{\frac{m^2}{4} + \frac{q^3}{27}}}. \tag{4}$$

If we suppose this root to be equal to $a$, the factors of (1) become $x^3 + qx + m = (x - a)(x^2 + ax + a^2 + q)$;

Therefore
$$x^2 + ax + a^2 + q = 0,$$
whence
$$x = \tfrac{1}{2}\left(-a \pm \sqrt{3a^2 - 4q}\right). \tag{5}$$

Equations (4) and (5) contain the roots of (1).

### EXAMPLES.

What are the roots of the equation $x^3 - 18x^2 + 101x = 180$?

By making the second term disappear, we have

$$y^3 - 7y = 6.$$

Then
$$q = -7 \text{ and } m = -6.$$

By substituting these values of $q$ and $m$ in equation (4), we have

$$x = \sqrt[3]{3 + \tfrac{10}{9}\sqrt{-3}} + \sqrt[3]{3 - \tfrac{10}{9}\sqrt{-3}}.$$

Now, each of these terms may be expanded by the binomial theorem, and added together. The terms involving $\sqrt{-3}$ will cancel, leaving $x$ a *real* quantity. This is called the IRREDUCIBLE CASE, and always arises when $\frac{m^2}{4} + \frac{q^3}{27}$ is a negative quantity. Its occurrence renders formula (4) entirely useless in practice. The roots of the given equation are 4, 5, and 9, and yet the formula will not reveal them without the use of an infinite series.

**272.** *Numerical Solution of Cubic Equations.*

Take the equation

$$x^3 + px^2 + qx = m. \qquad (1)$$

Find by trial a number which, on being substituted for $x$ in the given equation, will produce a result *less* than $m$, but such that if it is increased by *unity*, and again substituted, the result will be *greater* than $m$. Let $r$ be such a number. Then, if we regard it temporarily as the exact root, we may write

$$r^3 + pr^2 + qr = m; \qquad (2)$$

Whence
$$r = \frac{m}{q + pr + r^2}.$$

Having found $r$, denote the remaining figures of the root by $y$, whence

$$x = r + y. \qquad (3)$$

Substitute this value for $x$ in equation (1), and we have

$$(r + y)^3 + p\,(r + y)^2 + q\,(r + y) = m. \qquad (4)$$

Expand, and arrange in reference to $y$, and we have

$$y^3 + (3r + p)\,y^2 + (3r^2 + 2pr + q)\,y + (r^3 + pr^2 + qr) = m. \quad (5)$$

Make $\qquad p^1 = 3r + p,\ q^1 = 3r^2 + 2pr + q,$

and $\qquad m^1 = m - (r^3 + pr^2 + qr),$

and we have $\qquad y^3 + p^1y^2 + q^1y = m^1. \qquad (6)$

The first figure in the root of equation (6) which may be found in precisely the same way as in equation (1), is the second figure in the root of (1). By repeating the process, the third, fourth, fifth, &c. figures of the root of (1) may be found.

In applying the preceding principles, we proceed as follows:—

1. Arrange the coëfficients with their signs in a line, and to the right of them place the right-hand member of the equation.

2. Having found the first figure of the root, multiply it into the first coëfficient and add the product to the *second* coëfficient, which sum multiply by the same figure of the root, and add the product to the *third* coëfficient, multiply this sum by the same first figure and subtract the product from the term constituting the second member of the equation.

The remainder is the first dividend.

3. Multiply the first coëfficient by the same first figure of the root, and add the product to the last number under the *second* coëfficient; which sum must be multiplied by the same figure, and the product added to the last number under the *third* coëfficient.

This last sum is the first *trial divisor*.

Multiply the first coëfficient by the first figure of the root, and add the product to the last figure under the *second* coëfficient.

4. Divide the *first dividend* by the *first* trial divisor, and the quotient is the *second figure* of the root. With this figure proceed exactly as with the first figure, observing the rules for decimals, signs, &c.

5. After reaching the fourth or fifth decimal place, four or five other places may be found by dividing the last dividend by the last trial divisor.

## EXAMPLES.

1. Given $x^3 + x^2 + x = 100$, to find the values of $x$.

*Operation.*

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | 100 | 4.264429973 |
| | 5 | 21 | 84 | |
| | 9 | ×57 | *16 | |
| | 13.2 | 59.64 | 11.928 | |
| | 13.4 | ×62.32 | *4.072 | |
| | 13.66 | 63.1396 | 3.788376 | |
| | 13.72 | ×63.9628 | *.283624 | |
| | 13.784 | 64 017936 | 256071744 | |
| | 13.788 | ×64.073088 | *27552256 | |
| | 13.7924 | 64.0786496 | 25631441984 | |
| | 13.7928 | ×64.08412208 | *1920814016 | |
| | 13.7932 | | 1281682441 | |
| | | | 639131575 | |
| | | | 576757098 | |
| | | | 62374477 | |
| | | | 57675709 | |
| | | | 4698768 | |
| | | | 4485888 | |
| | | | 212880 | |
| | | | 192252 | |

The numbers marked × are divisors; those marked * are corresponding dividends. The reason for a simple division after reaching the fifth figure is apparent.

The other roots are now obtained as follows:—

Divide $x^3 + x^2 + x - 100 = 0$ by $x - 4.264429973$, and we have $$x^2 + 5.264429973x = -23.449792962;$$

Whence $$x = -2.632214986 \pm 4.063402165\sqrt{-1}.$$

2. Find a root of the equation $x^3 + 10x^2 + 5x = 2600$.

*Operation.*

| | | | | |
|---|---|---|---|---|
| 1 | 10 | 5 | 2600 | 11.0067993399 |
| | 21 | 236 | 2596 | |
| | 32 | 588 | 4 | |
| | 43.006 | 588.258036 | 3.529548216 | |
| | 43.016 | 588.516132 | 470451784 | |
| | 43.0227 | 588.54624789 | 411982373523 | |
| | 43.0234 | 588.57636427 | 58469410477 | |
| | 43.02419 | 588.5802364471 | 52972221280 | |
| | 43.02428 | 588.5841086323 | 5497189196 | |
| | | | 529722 | |
| | | | 19996 | |
| | | | 17657 | |
| | | | 2339 | |
| | | | 1765 | |
| | | | 574 | |
| | | | 522 | |
| | | | 52 | |
| | | | 45 | |
| | | | 7 | |

Here we commence with 11, and the process is precisely the same as in the preceding example. In the division only the constant part of the divisor need be considered, and the corresponding part of the dividend.

3. Find $x$ in the equation $x^3 - 2x = 5$.

This equation is the same as $x^3 \pm 0x^2 - 2x = 5$.

*Operation.*

| | | | |
|---|---|---|---|
| 1 | $\pm 0$ | $-2$ | 5 \| 2.094551482 |
| | 2 | 2 | 4 |
| | 4 | 10 | 1.000000 |
| | 6.09 | 10.5481 | 949329 |
| | 6.18 | 11.1043 | 50671 |
| | 6.274 | 11.129396 | 44517584 |
| | 6.278 | 11.154508 | 6153416 |
| | 6 2825 | 11.15764925 | 5578824625 |
| | 6.2830 | 11.16079075 | 574591375 |
| | 6.28305 | 11.1611049025 | 558055245 |
| | 6.28310 | 11.1614190575 | 16536 |
| | | | 11161 |
| | | | 5375 |
| | | | 4464 |
| | | | 911 |
| | | | 882 |
| | | | 29 |
| | | | 22 |
| | | | 7 |

4. Find $x$ in the equation $x^3 - x^2 - 15x = -24$.

*Operation.*

| | | | | |
|---|---|---|---|---|
| 1 | − 1 | − 15 | − 24 | 1.7550451874 |
| | 0 | − 15 | − 15 | |
| | 1 | − 14 | − 9 | |
| | 2.7 | − 12.11 | − 8.477 | |
| | 3.4 | − 9.73 | − .523 | |
| | 4.15 | − 9.5225 | − .476125 | |
| | 4.20 | − 9.3125 | − 46875 | |
| | 4.255 | − 9.291225 | − 46456125 | |
| | 4.260 | − 9.269925 | − 418875 | |
| | 4.26504 | − 9.269754 | − 370790175936 | |
| | 4.26508 | − 9.269584 | − 48084824 | |
| | | | − 463475 | |
| | | | − 17373 | |
| | | | − 9269 | |
| | | | − 8104 | |
| | | | − 7415 | |
| | | | − 689 | |
| | | | − 648 | |
| | | | − 41 | |
| | | | − 36 | |

5. Find the roots of the equation $x^3 - 1242x^2 + 9858x = -17276$.

*Operation.*

| | | | | |
|---|---|---|---|---|
| 1 | $-1242$ | $9858$ | $-17276$ | $1000 + 200 + 30 + 4 = 1234.$ |
| | $-242$ | $-232142$ | $-232142000$ | |
| | $758$ | $525858$ | $232124724$ | |
| | $1758$ | $917458$ | $183491600$ | |
| | $1958$ | $1349058$ | $48623124$ | |
| | $2158$ | $1420698$ | $42620940$ | |
| | $2358$ | $1493238$ | $6002184$ | |
| | $2388$ | $1503046$ | $6002184$ | |
| | $2418$ | | | |
| | $2448$ | | | |
| | $2452$ | | | |

Divide the given equation by $x - 1234$, and we have

$$x^2 - 8x = 14;$$

Whence $x = 9.477225$, or $-1.477225$.

6. Given $13x^3 + \frac{3}{8}x^2 - 100\frac{1}{4}x = -12\frac{1}{2}$, to find the values of $x$.

*Operation.*

| | | | | |
|---|---|---|---|---|
| 13 | $.375$ | $-100.25$ | $-12.5$ | $.125 = \frac{1}{8}$ |
| | $1.675$ | $-100.0825$ | $-10.00825$ | |
| | $2.975$ | $-99.785$ | $-2.49175$ | |
| | $4.275$ | $-99.6943$ | $-1.993886$ | |
| | $4.535$ | $-99.5984$ | $-.497864$ | |
| | $4.795$ | $-99.5728$ | $-.497864$ | |
| | $5.055$ | | | |
| | $5.120$ | | | |

Divide the given equation by $x - \frac{1}{8}$, and we have

$$13x^2 + 2x = 100;$$

Whence $x = 2.6975$, or $-2.8514$.

7. Given $x^3 - 15x^2 + 63x = 50$, to find the values of $x$.

*Ans.* $x = 1.028039231$, $x = 6.576535$, and $x = 7.395426$.

8. Find a root of the equation $x^3 + x^2 = 500$.

*Ans.* $x = 7.61727975$, &c.

9. Find a root of the equation $x^3 + x = 500$.

*Ans.* $x = 7.89500828$, &c.

10. Find a root of the equation $x^3 + 2x^2 + 3x = 13089030$.

*Ans.* $x = 235$.

HIGHER EQUATIONS.

**273.** The same method may be pursued in solving a numerical equation of any degree whatever.

1. Given $x^4 + 4x^3 + 3x^2 + 2x = 1400$, to find $x$.

*Operation.*

| | | | | | |
|---|---|---|---|---|---|
| 1 | 4 | 3 | 2 | 1400 | 5.216114541, &c. |
| | 9 | 48 | 242 | 1210 | |
| | 14 | 118 | ×834 | *190 | |
| | 19 | 213 | 875.568 | 175.1136 | |
| | 24.2 | 217.84 | ×920.112 | *14.8864 | |
| | 24.4 | 222.72 | 922.390881 | 9.22390881 | |
| | 24.6 | 227.64 | ×924.672244 | *5.66249119 | |
| | 24.81 | 227.8881 | 926.043446 | 5.55626067 | |
| | 24.82 | 228.1363 | ×927.415 | *.10623052 | |
| | 24.83 | 228.3843 . | 927.43 | 9274384 | |
| | 24 | 228 | ×927.4 | *134866 | |
| | 24 | 228. | | 92746 | |
| | 24 | 228 | | 42120 | |
| | 24 | 228 | | 37098 | |
| | 24 | 228 | | 5021 | |
| | | | | 4637 | |
| | | | | 384 | |
| | | | | 370 | |
| | | | | 14 | |
| | | | | 9 | |
| | | | | 5 | |

We have first arranged the coëfficients and second member of the equation in a line. We find by trial the first figure of the root to be 5. The work then proceeds thus:—

$1 \times 5 + 4 = 9$ in the first column, $5 \times 9 + 3 = 48$ in the second column, $48 \times 5 + 2 = 242$ in the third column, $242 \times 5 = 1210$, the first subtrahend.

$1 \times 5 + 9 = 14$ in the first column, $14 \times 5 + 48 = 118$ in the second column, $118 \times 5 + 242 = 832$, the first *trial divisor.*

$1 \times 5 + 14 = 19$ in the first column, and $19 \times 5 + 118 = 213$ in the second column.

$1 \times 5 + 19 = 24.$

We now find the quotient of the first dividend by the trial divisor to be .2. Then,

$1 \times .2 + 24 = 24.2$ in first column, $24.2 \times .2 + 213 = 217.84$ in second column, $217.84 \times .2 + 832 = 875.568$ in third column, and $875.568 \times .2 = 175.1136$, the second subtrahend.

Then $1 \times .2 + 24.2 = 24.4$ in first column, $24.4 \times .2 + 217.84 = 222.72.$

$222.72 \times .2 + 875.568 = 920.112$, the second *trial divisor.*

$1 \times .2 + 24.4 = 24.6$ in first column, $24.6 \times .2 + 222.72 = 227.64.$

$1 \times .2 + 24.6 = 24.8.$

We now find the quotient of the dividend by the second trial divisor to be .01. With this proceed exactly as with the others.

As the decimals on the right do not affect the result when a limited number of figures is sought, we may disregard them in the calculation.

On arriving at the fourth decimal place, we simply divide for the others.

To this equation we may find another root, thus:—

By trial, $-7$ is found to be the first figure, when we proceed as follows:—

| | | | | | |
|---|---|---|---|---|---|
| 1 | 4 | 3 | 2 | 1400 | − 7.26948592 |
| | − 3 | 24 | − 166 | 1162 | |
| | − 10 | 94 | − 824 | 238 | |
| | − 17 | 213 | − 867.568 | 173.5136 | |
| | − 24.2 | 217.84 | − 912.112 | 64.4864 | |
| | − 24.4 | 222.72 | − 925.859896 | 55.5515937 | |
| | − 24.6 | 227.64 | − 939.697504 | 8.9348063 | |
| | − 24.86 | 229.1316 | − 941.78866 | 8.4760979 | |
| | − 24.92 | 230.6268 | − 943.89185 | 4587084 | |
| | − 24.98 | 232.1256 | − 943.9849 | 3775939 | |
| | − 25.049 | 232.3510 | − 944.078 | 81114 | |
| | − 25.058 | 232.5765 | | 75525 | |
| | − 25.067 | 232.8021 | | 5589 | |
| | − 25.0764 | 232.81 | | 4720 | |
| | − 25.0768 | 232 | | 869 | |
| | | | | 846 | |
| | | | | 23 | |
| | | | | 18 | |
| | | | | 5 | |

The work in this example will be readily followed. The other two roots might be found by dividing the given equation by $x + 7.26948592$, and then by $x - 5.216114541$, thus reducing it to a quadratic equation, to be solved in the usual way.

2. Find one root of the equation $x^4 + 4x^3 - 4x^2 - 11x = -4$.

*Ans.* $x = 1.63691356$, &c.

3. Find one root of the equation $x^4 - 3x^2 + 75x = 10000$.

*Ans.* 9.88600270094, &c.

## QUESTIONS FOR EXAMINATION.

1. What is the numerical value of $\frac{x^{3n}}{x^n - 1} - \frac{x^{2n}}{x^n + 1} + \frac{1}{x^n + 1} - \frac{1}{x^n - 1}$ when $n = 1$ and $x = 1, 2, 3, 4, 5,$ &c.?

2. What is the numerical value of $a^{-1}\left(1 + \frac{x}{a}\right)^{-1}$ when $a = 1, 2, 3, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4},$ &c., and $x = 1, 2, 3, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4},$ &c.?

3. What is the value of $a^{\frac{4}{3}}\left(1 - \frac{x^2}{a^2}\right)^{\frac{2}{3}}$ when $a = \sqrt[4]{8}$ and $x = 0$? when $a = 3$ and $x = 1$? $a = 6$ and $x = 3$?

4. What is the value of $a^{-1}\left(1 + \frac{x^4}{a^4}\right)^{\frac{1}{4}}$ when $a = 2$ and $x = 0$? when $a = 2$ and $x = \sqrt[4]{65}$?

5. Add together $\frac{x^2 + x - 20}{x^2 - 7x + 12}$ and $\frac{x^2 + 2x - 15}{x^2 + x - 20}$.

6. From $\frac{x^2 + x - 20}{x^2 - 7x + 12}$ take $\frac{x^2 + 2x - 15}{x^2 + x - 20}$.

7. Multiply $\frac{x^2 + x - 20}{x^2 - 7x + 12}$ by $\frac{x^2 + 2x - 15}{x^2 + x - 20}$.

8. Divide $\frac{x^2 + x - 20}{x^2 - 7x + 12}$ by $\frac{x^2 + 2x - 15}{x^2 + x - 20}$.

9. Find the value of $x$ in the equation

$$\frac{2x + 1}{29} - 33\tfrac{1}{2} + \frac{x}{4} = 3x - 226\tfrac{1}{2}.$$

10. Find the value of $x$ in the equation

$$23 + \frac{5x - 1}{11} + \frac{3x - 2}{5} - \frac{11x - 3}{12} = \frac{13x}{3} - 5 - \frac{8x - 2}{7}.$$

11. Given $\frac{x + ax - bx}{a - b} = \frac{cx - d}{c}$, to find $x$.

12. Given $a^2 + b^2 + a^2x - b^2x = \frac{a^2 + ab + b^2}{a^2 - b^2}$, to find $x$.

13. Given $x + \frac{y}{41} = 45$ and $\frac{x}{41} + y = 165$, to find $x$ and $y$.

*Ans.* $x = 41$, $y = 164$.

14. Given $\frac{3}{x} + \frac{4}{y} = 4$ and $\frac{6}{x} + \frac{10}{y} = 9$, to find $x$ and $y$.

*Ans.* $x = 1\frac{1}{2}$; $y = 2$.

15. Given $x + \frac{y}{a} = b$ and $\frac{x}{c} + y = d$, to find $x$ and $y$.

*Ans.* $x = \frac{abc - cd}{ac - 1}$, $y = \frac{acd - ab}{ac - 1}$.

16. Given $\left\{\begin{array}{l} x + a(y + z + w) = m, \\ y + b(x + y + w) = n, \\ z + c(x + y + w) = p, \\ w + d(x + y + z) = q, \end{array}\right\}$ to find $x$, $y$, $z$, and $w$.

$$Ans.\ x = \frac{m}{1-a} - \frac{a}{1-a}\left(\frac{\frac{m}{1-a} + \frac{n}{1-b} + \frac{p}{1-c} + \frac{q}{1-d}}{1 + \frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} + \frac{d}{1-d}}\right)$$

17. Given $\frac{(x+5)^2}{52} = \frac{9x^2}{117}$, to find the value of $x$. *Ans.* $x = 5$.

18. Given $\frac{x^2 + a}{2x^2 - b} = \frac{2x^2 - b}{x^2 + a}$, to find the values of $x$.

*Ans.* $x = \pm\sqrt{a + b}$.

19. Given $x^2 - x = 56$, to find the values of $x$.

*Ans.* $x = 8$ and $-7$.

20. Given $\frac{x + 11}{x} + \frac{9 + 4x}{x^2} = 7$, to find the values of $x$.

*Ans.* $x = 3$ or $-\frac{1}{2}$.

21. Given $16x - x^2 = 65$, to find the values of $x$.

*Ans.* $x = 8 \pm \sqrt{-1}$.

22. Given $x^2 - \frac{1617x}{21} = -\frac{20748}{21}$, to find the values of $x$.

*Ans.* $x = 60.72$ or $16.27$.

23. Given $x - x^{\frac{1}{2}} = 20$, to find the values of $x$. *Ans.* $x = 25$.

24. Given $(x + 10)^{\frac{1}{2}} - (x + 10)^{\frac{1}{4}} = 2$, to find $x$. *Ans.* $x = 6$.

25. Given $(7x^2 - x + 1)^2 - (7x^2 - x + 1) = 702$, to find the values of $x$. *Ans.* $x = \frac{1}{14}(1 \pm \sqrt{-755})$, $x = 2$ or $-\frac{13}{7}$.

26. Given $x^2 + 2x = 23$, to find $x$.

*Ans.* $x = 3.8989795$ and $x = -5.8989795$.

27. Given $\dfrac{1 + \dfrac{x+1}{x-1}}{1 - \dfrac{x-1}{x+1}} = 7\frac{1}{2}$. *Ans.* $x = 5$.

28. Given $\dfrac{x + y - \dfrac{2xy}{x+y}}{x + y - \dfrac{x^2 + y^2}{x+y}} = 1$ and $7xy = 28$, to find $x$ and $y$.

*Ans.* $x = 2$, $y = 2$.

29. Given $\dfrac{\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y}}{\dfrac{x+y}{x-y} - \dfrac{x-y}{x+y}} = a$ and $xy = b$, to find $x$ and $y$.

*Ans.* $y = \frac{1}{2}\left[\pm\sqrt{ab + 2b} \mp \sqrt{ab - 2b}\right]$,

$x = \frac{1}{2}\left[\pm\sqrt{ab + 2b} \pm \sqrt{ab - 2b}\right]$.

30. Given $x + y + \sqrt{x + y} = 6$ and $x^2 + y^2 = 10$, to find $x$ and $y$.

*Ans.* $x = 3$ or $1$, or $4\frac{1}{2} \pm \frac{1}{2}\sqrt{-61}$,

$y = 1$ or $3$, or $4\frac{1}{2} \mp \frac{1}{2}\sqrt{-61}$.

31. Given $\dfrac{x^2 + 6x + 8}{x^2 - x - 6} - \dfrac{x^2 + 3x - 40}{x^2 + x - 30} = 1\frac{9}{35}$, to find $x$.

*Ans.* $x = 8$, or $-7\frac{1}{11}$.

32. Given $\dfrac{x^2 + 1}{x^2 - 1} - \dfrac{x^2 - 1}{x^2 + 1} = \frac{9}{20}$, to find $x$.

*Ans.* $x = \pm 3$, or $x = \pm\frac{1}{3}\sqrt{-1}$.

33. One root of $x^3 - 13\frac{1}{2}x^2 + 46\frac{1}{2}x = 20$ is $8$: what are the other roots? *Ans.* $x = 5$ and $x = \frac{1}{2}$.

34. One root of $x^3 + 5\frac{1}{6}x^2 - 4\frac{5}{6}x = -1$ is $-6$: what are the other roots? *Ans.* $\frac{1}{2}$ and $\frac{1}{3}$

35. Has the equation $x^3 + 2x^2 - 15x = 36$ equal roots? If so, find all the roots.

*Ans.* It has; and $-3$, $-3$, and $-4$ are the roots.

36. What are the roots of the equation $x^4 - 20x^3 + 142x^2 - 420x = -441$? *Ans.* 7, 7, 3, and 3.

37. Find one value of $x$ in the equation $x^3 - 2x = 50$.

*Ans.* 3.8648854.

38. If a certain number is divided by 7, and if, then, the quotient is taken from the sum of the dividend and divisor, the remainder will be 73: what is the number? *Ans.* 77.

39. To find three numbers in arithmetical progression, of which the first is to the third as 5 to 9, and the sum of all three is 63.

*Ans.* 15, 21, 27.

40. A sets out from C toward D, and travels 8 miles a day. After he had gone 27 miles, B set out from D toward C, and goes every day $\frac{1}{20}$th of the whole journey, and after he had traveled as many days as he goes miles in one day, he met A. Required the distance of the place C from D. *Ans.* 180 or 60 miles.

41. Two post-boys, A and B, set out at the same time from two cities, 500 miles apart, in order to meet each other. A rides 60 miles the first day, 55 the second, 50 the third, and so on, decreasing 5 miles every day. B goes 40 miles the first day, 45 the second, 50 the third, and so on, increasing 5 miles every day. In what number of days will they meet? *Ans.* In 5 days.

42. A tree, 100 feet high, stands just at the water-line on the bank of a river 200 feet wide. The tree broke in a gale of wind, and the upper part was found to point exactly to the water-line of the opposite bank, the top being within 20 feet of the surface of the water. How high from the ground did the tree break?

*Ans.* 30.472 feet.

| No. | Square Root. | No. | Square Root. | No. | Square Root. |
|---|---|---|---|---|---|
| 1 | 1·0000000 | 61 | 7·8102497 | 121 | 11·0000000 |
| 2 | 1·4142136 | 62 | 7·8740079 | 122 | 11·0453610 |
| 3 | 1·7320508 | 63 | 7·9372539 | 123 | 11·0905365 |
| 4 | 2·0000000 | 64 | 8·0000000 | 124 | 11·1355287 |
| 5 | 2·2360680 | 65 | 8·0622577 | 125 | 11·1803399 |
| 6 | 2·4494897 | 66 | 8·1240384 | 126 | 11·2249722 |
| 7 | 2·6457513 | 67 | 8·1853528 | 127 | 11·2694277 |
| 8 | 2·8284271 | 68 | 8·2462113 | 128 | 11·3137085 |
| 9 | 3·0000000 | 69 | 8·3066239 | 129 | 11·3578167 |
| 10 | 3·1622777 | 70 | 8·3666003 | 130 | 11·4017543 |
| 11 | 3·3166248 | 71 | 8·4261498 | 131 | 11·4455231 |
| 12 | 3·4641016 | 72 | 8·4852814 | 132 | 11·4891253 |
| 13 | 3·6055513 | 73 | 8·5440037 | 133 | 11·5325626 |
| 14 | 3·7416574 | 74 | 8·6023253 | 134 | 11·5758369 |
| 15 | 3·8729833 | 75 | 8·6602540 | 135 | 11·6189500 |
| 16 | 4·0000000 | 76 | 8·7177979 | 136 | 11·6619038 |
| 17 | 4·1231056 | 77 | 8·7749644 | 137 | 11·7046999 |
| 18 | 4·2426407 | 78 | 8·8317609 | 138 | 11·7473444 |
| 19 | 4·3588989 | 79 | 8·8881944 | 139 | 11·7898261 |
| 20 | 4·4721360 | 80 | 8·9442719 | 140 | 11·8321596 |
| 21 | 4·5825757 | 81 | 9·0000000 | 141 | 11·8743421 |
| 22 | 4·6904158 | 82 | 9·0553851 | 142 | 11·9163753 |
| 23 | 4·7958315 | 83 | 9·1104336 | 143 | 11·9582607 |
| 24 | 4·8989795 | 84 | 9·1651514 | 144 | 12·0000000 |
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| 172 | 5528 | 5781 | 6033 | 6285 | 6537 | 6789 | 7041 | 7292 | 7544 | 7795 | 252 |
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| 285 | 4845 | 4997 | 5150 | 5302 | 5454 | 5606 | 5758 | 5910 | 6062 | 6214 | 152 |
| 286 | 6366 | 6518 | 6670 | 6821 | 6973 | 7125 | 7276 | 7428 | 7579 | 7731 | 152 |
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| 291 | 3893 | 4042 | 4191 | 4340 | 4490 | 4639 | 4788 | 4936 | 5085 | 5234 | 149 |
| 292 | 5383 | 5532 | 5680 | 5829 | 5977 | 6126 | 6274 | 6423 | 6571 | 6719 | 149 |
| 293 | 6868 | 7016 | 7164 | 7312 | 7460 | 7608 | 7756 | 7904 | 8052 | 8200 | 148 |
| 294 | 8347 | 8495 | 8643 | 8790 | 8938 | 9085 | 9233 | 9380 | 9527 | 9675 | 148 |
| 295 | 9822 | 9969 | •116 | •263 | •410 | •557 | •704 | •851 | •998 | 1145 | 147 |
| 296 | 471292 | 1438 | 1585 | 1732 | 1878 | 2025 | 2171 | 2318 | 2464 | 2610 | 146 |
| 297 | 2756 | 2903 | 3049 | 3195 | 3341 | 3487 | 3633 | 3779 | 3925 | 4071 | 146 |
| 298 | 4216 | 4362 | 4508 | 4653 | 4799 | 4944 | 5090 | 5235 | 5381 | 5526 | 146 |
| 299 | 5671 | 5816 | 5962 | 6107 | 6252 | 6397 | 6542 | 6687 | 6832 | 6976 | 145 |
| 300 | 477121 | 7266 | 7411 | 7555 | 7700 | 7844 | 7989 | 8133 | 8278 | 8422 | 145 |
| 301 | 8566 | 8711 | 8855 | 8999 | 9143 | 9287 | 9431 | 9575 | 9719 | 9863 | 144 |
| 302 | 480007 | 0151 | 0294 | 0438 | 0582 | 0725 | 0869 | 1012 | 1156 | 1299 | 144 |
| 303 | 1443 | 1586 | 1729 | 1872 | 2016 | 2159 | 2302 | 2445 | 2588 | 2731 | 143 |
| 304 | 2874 | 3016 | 3159 | 3302 | 3445 | 3587 | 3730 | 3872 | 4015 | 4157 | 143 |
| 305 | 4300 | 4442 | 4585 | 4727 | 4869 | 5011 | 5153 | 5295 | 5437 | 5579 | 142 |
| 306 | 5721 | 5863 | 6005 | 6147 | 6289 | 6430 | 6572 | 6714 | 6855 | 6997 | 142 |
| 307 | 7138 | 7280 | 7421 | 7563 | 7704 | 7845 | 7986 | 8127 | 8269 | 8410 | 141 |
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| 311 | 2760 | 2900 | 3040 | 3179 | 3319 | 3458 | 3597 | 3737 | 3876 | 4015 | 139 |
| 312 | 4155 | 4294 | 4433 | 4572 | 4711 | 4850 | 4989 | 5128 | 5267 | 5406 | 139 |
| 313 | 5544 | 5683 | 5822 | 5960 | 6099 | 6238 | 6376 | 6515 | 6653 | 6791 | 139 |
| 314 | 6930 | 7068 | 7206 | 7344 | 7483 | 7621 | 7759 | 7897 | 8035 | 8173 | 138 |
| 315 | 8311 | 8448 | 8586 | 8724 | 8862 | 8999 | 9137 | 9275 | 9412 | 9550 | 138 |
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| 320 | 505150 | 5286 | 5421 | 5557 | 5693 | 5828 | 5964 | 6099 | 6234 | 6370 | 136 |
| 321 | 6505 | 6640 | 6776 | 6911 | 7046 | 7181 | 7316 | 7451 | 7586 | 7721 | 135 |
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| 327 | 4548 | 4681 | 4813 | 4946 | 5079 | 5211 | 5344 | 5476 | 5609 | 5741 | 133 |
| 328 | 5874 | 6006 | 6139 | 6271 | 6403 | 6535 | 6668 | 6800 | 6932 | 7064 | 132 |
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| 335 | 5045 | 5174 | 5304 | 5434 | 5563 | 5693 | 5822 | 5951 | 6081 | 6210 | 129 |
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| 352 | 6543 | 6666 | 6789 | 6913 | 7036 | 7159 | 7282 | 7405 | 7529 | 7652 | 123 |
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| 360 | 556303 | 6423 | 6544 | 6664 | 6785 | 6905 | 7026 | 7146 | 7267 | 7387 | 120 |
| 361 | 7507 | 7627 | 7748 | 7868 | 7988 | 8108 | 8228 | 8349 | 8469 | 8589 | 120 |
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| 363 | 9907 | ••26 | •146 | •265 | •385 | •504 | •624 | •743 | •863 | •982 | 119 |
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| 365 | 2293 | 2412 | 2531 | 2650 | 2769 | 2887 | 3006 | 3125 | 3244 | 3362 | 119 |
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| 367 | 4666 | 4784 | 4903 | 5021 | 5139 | 5257 | 5376 | 5494 | 5612 | 5730 | 118 |
| 368 | 5848 | 5966 | 6084 | 6202 | 6320 | 6437 | 6555 | 6673 | 6791 | 6909 | 118 |
| 369 | 7026 | 7144 | 7262 | 7379 | 7497 | 7614 | 7732 | 7849 | 7967 | 8084 | 118 |
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| 371 | 9374 | 9491 | 9608 | 9725 | 9842 | 9959 | ••76 | •193 | •309 | •426 | 117 |
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| 373 | 1709 | 1825 | 1942 | 2058 | 2174 | 2291 | 2407 | 2523 | 2639 | 2755 | 116 |
| 374 | 2872 | 2988 | 3104 | 3220 | 3336 | 3452 | 3568 | 3684 | 3800 | 3915 | 116 |
| 375 | 4031 | 4147 | 4263 | 4379 | 4494 | 4610 | 4726 | 4841 | 4957 | 5072 | 116 |
| 376 | 5188 | 5303 | 5419 | 5534 | 5650 | 5765 | 5880 | 5996 | 6111 | 6226 | 115 |
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| 382 | 2063 | 2177 | 2291 | 2404 | 2518 | 2631 | 2745 | 2858 | 2972 | 3085 | 114 |
| 383 | 3199 | 3312 | 3426 | 3539 | 3652 | 3765 | 3879 | 3992 | 4105 | 4218 | 113 |
| 384 | 4331 | 4444 | 4557 | 4670 | 4783 | 4896 | 5009 | 5122 | 5235 | 5348 | 113 |
| 385 | 5461 | 5574 | 5686 | 5799 | 5912 | 6024 | 6137 | 6250 | 6362 | 6475 | 113 |
| 386 | 6587 | 6700 | 6812 | 6925 | 7037 | 7149 | 7262 | 7374 | 7486 | 7599 | 112 |
| 387 | 7711 | 7823 | 7935 | 8047 | 8160 | 8272 | 8384 | 8496 | 8608 | 8720 | 112 |
| 388 | 8832 | 8944 | 9056 | 9167 | 9279 | 9391 | 9503 | 9615 | 9726 | 9838 | 112 |
| 389 | 9950 | ••61 | •173 | •284 | •396 | •507 | •619 | •730 | •842 | •953 | 112 |
| 390 | 591065 | 1176 | 1287 | 1399 | 1510 | 1621 | 1732 | 1843 | 1955 | 2066 | 111 |
| 391 | 2177 | 2288 | 2399 | 2510 | 2621 | 2732 | 2843 | 2954 | 3064 | 3175 | 111 |
| 392 | 3286 | 3397 | 3508 | 3618 | 3729 | 3840 | 3950 | 4061 | 4171 | 4282 | 111 |
| 393 | 4393 | 4503 | 4614 | 4724 | 4834 | 4945 | 5055 | 5165 | 5276 | 5386 | 110 |
| 394 | 5496 | 5606 | 5717 | 5827 | 5937 | 6047 | 6157 | 6267 | 6377 | 6487 | 110 |
| 395 | 6597 | 6707 | 6817 | 6927 | 7037 | 7146 | 7256 | 7366 | 7476 | 7586 | 110 |
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| 401 | 3144 | 3253 | 3361 | 3469 | 3577 | 3686 | 3794 | 3902 | 4010 | 4118 | 108 |
| 402 | 4226 | 4334 | 4442 | 4550 | 4658 | 4766 | 4874 | 4982 | 5089 | 5197 | 108 |
| 403 | 5305 | 5413 | 5521 | 5628 | 5736 | 5844 | 5951 | 6059 | 6166 | 6274 | 108 |
| 404 | 6381 | 6489 | 6596 | 6704 | 6811 | 6919 | 7026 | 7133 | 7241 | 7348 | 107 |
| 405 | 7455 | 7562 | 7669 | 7777 | 7884 | 7991 | 8098 | 8205 | 8312 | 8419 | 107 |
| 406 | 8526 | 8633 | 8740 | 8847 | 8954 | 9061 | 9167 | 9274 | 9381 | 9488 | 107 |
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| 410 | 612784 | 2890 | 2996 | 3102 | 3207 | 3313 | 3419 | 3525 | 3630 | 3736 | 106 |
| 411 | 3842 | 3947 | 4053 | 4159 | 4264 | 4370 | 4475 | 4581 | 4686 | 4792 | 106 |
| 412 | 4897 | 5003 | 5108 | 5213 | 5319 | 5424 | 5529 | 5634 | 5740 | 5845 | 105 |
| 413 | 5950 | 6055 | 6160 | 6265 | 6370 | 6476 | 6581 | 6686 | 6790 | 6895 | 105 |
| 414 | 7000 | 7105 | 7210 | 7315 | 7420 | 7525 | 7629 | 7734 | 7839 | 7943 | 105 |
| 415 | 8048 | 8153 | 8257 | 8362 | 8466 | 8571 | 8676 | 8780 | 8884 | 8989 | 105 |
| 416 | 9093 | 9198 | 9302 | 9406 | 9511 | 9615 | 9719 | 9824 | 9928 | ••32 | 104 |
| 417 | 620136 | 0240 | 0344 | 0448 | 0552 | 0656 | 0760 | 0864 | 0968 | 1072 | 104 |
| 418 | 1176 | 1280 | 1384 | 1488 | 1592 | 1695 | 1799 | 1903 | 2007 | 2110 | 104 |
| 419 | 2214 | 2318 | 2421 | 2525 | 2628 | 2732 | 2835 | 2939 | 3042 | 3146 | 104 |
| 420 | 623249 | 3353 | 3456 | 3559 | 3663 | 3766 | 3869 | 3973 | 4076 | 4179 | 103 |
| 421 | 4282 | 4385 | 4488 | 4591 | 4695 | 4798 | 4901 | 5004 | 5107 | 5210 | 103 |
| 422 | 5312 | 5415 | 5518 | 5621 | 5724 | 5827 | 5929 | 6032 | 6135 | 6238 | 103 |
| 423 | 6340 | 6443 | 6546 | 6648 | 6751 | 6853 | 6956 | 7058 | 7161 | 7263 | 103 |
| 424 | 7366 | 7468 | 7571 | 7673 | 7775 | 7878 | 7980 | 8082 | 8185 | 8287 | 102 |
| 425 | 8389 | 8491 | 8593 | 8695 | 8797 | 8900 | 9002 | 9104 | 9206 | 9308 | 102 |
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| 428 | 1444 | 1545 | 1647 | 1748 | 1849 | 1951 | 2052 | 2153 | 2255 | 2356 | 101 |
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| 430 | 633468 | 3569 | 3670 | 3771 | 3872 | 3973 | 4074 | 4175 | 4276 | 4376 | 100 |
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| 442 | 5422 | 5521 | 5619 | 5717 | 5815 | 5913 | 6011 | 6110 | 6208 | 6306 | 98 |
| 443 | 6404 | 6502 | 6600 | 6698 | 6796 | 6894 | 6992 | 7089 | 7187 | 7285 | 98 |
| 444 | 7383 | 7481 | 7579 | 7676 | 7774 | 7872 | 7969 | 8067 | 8165 | 8262 | 98 |
| 445 | 8360 | 8458 | 8555 | 8653 | 8750 | 8848 | 8945 | 9043 | 9140 | 9237 | 97 |
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| 455 | 8011 | 8107 | 8202 | 8298 | 8393 | 8488 | 8584 | 8679 | 8774 | 8870 | 95 |
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| 458 | 660865 | 0960 | 1055 | 1150 | 1245 | 1339 | 1434 | 1529 | 1623 | 1718 | 95 |
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| 473 | 4861 | 4953 | 5045 | 5137 | 5228 | 5320 | 5412 | 5503 | 5595 | 5687 | 92 |
| 474 | 5778 | 5870 | 5962 | 6053 | 6145 | 6236 | 6328 | 6419 | 6511 | 6602 | 92 |
| 475 | 6694 | 6785 | 6876 | 6968 | 7059 | 7151 | 7242 | 7333 | 7424 | 7516 | 91 |
| 476 | 7607 | 7698 | 7789 | 7881 | 7972 | 8063 | 8154 | 8245 | 8336 | 8427 | 91 |
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| 484 | 4845 | 4935 | 5025 | 5114 | 5204 | 5294 | 5383 | 5473 | 5563 | 5652 | 90 |
| 485 | 5742 | 5831 | 5921 | 6010 | 6100 | 6189 | 6279 | 6368 | 6458 | 6547 | 89 |
| 486 | 6636 | 6726 | 6815 | 6904 | 6994 | 7083 | 7172 | 7261 | 7351 | 7440 | 89 |
| 487 | 7529 | 7618 | 7707 | 7796 | 7886 | 7975 | 8064 | 8153 | 8242 | 8331 | 89 |
| 488 | 8420 | 8509 | 8598 | 8687 | 8776 | 8865 | 8953 | 9042 | 9131 | 9220 | 89 |
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| 491 | 1081 | 1170 | 1258 | 1347 | 1435 | 1524 | 1612 | 1700 | 1789 | 1877 | 88 |
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| 494 | 3727 | 3815 | 3903 | 3991 | 4078 | 4166 | 4254 | 4342 | 4430 | 4517 | 88 |
| 495 | 4605 | 4693 | 4781 | 4868 | 4956 | 5044 | 5131 | 5219 | 5307 | 5394 | 88 |
| 496 | 5482 | 5569 | 5657 | 5744 | 5832 | 5919 | 6007 | 6094 | 6182 | 6269 | 87 |
| 497 | 6356 | 6444 | 6531 | 6618 | 6706 | 6793 | 6880 | 6968 | 7055 | 7142 | 87 |
| 498 | 7229 | 7317 | 7404 | 7491 | 7578 | 7665 | 7752 | 7839 | 7926 | 8014 | 87 |
| 499 | 8101 | 8188 | 8275 | 8362 | 8449 | 8535 | 8622 | 8709 | 8796 | 8883 | 87 |
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| 501 | 9838 | 9924 | ••11 | ••98 | •184 | •271 | •358 | •444 | •531 | •617 | 87 |
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| 523 | 8502 | 8585 | 8668 | 8751 | 8834 | 8917 | 9000 | 9083 | 9165 | 9248 | 83 |
| 524 | 9331 | 9414 | 9497 | 9580 | 9663 | 9745 | 9828 | 9911 | 9994 | ••77 | 83 |
| 525 | 720159 | 0242 | 0325 | 0407 | 0490 | 0573 | 0655 | 0738 | 0821 | 0903 | 83 |
| 526 | 0986 | 1068 | 1151 | 1233 | 1316 | 1398 | 1481 | 1563 | 1646 | 1728 | 82 |
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| 532 | 5912 | 5993 | 6075 | 6156 | 6238 | 6320 | 6401 | 6483 | 6564 | 6646 | 82 |
| 533 | 6727 | 6809 | 6890 | 6972 | 7053 | 7134 | 7216 | 7297 | 7379 | 7460 | 81 |
| 534 | 7541 | 7623 | 7704 | 7785 | 7866 | 7948 | 8029 | 8110 | 8191 | 8273 | 81 |
| 535 | 8354 | 8435 | 8516 | 8597 | 8678 | 8759 | 8841 | 8922 | 9003 | 9084 | 81 |
| 536 | 9165 | 9246 | 9327 | 9408 | 9489 | 9570 | 9651 | 9732 | 9813 | 9893 | 81 |
| 537 | 9974 | ••55 | •136 | •217 | •298 | •378 | •459 | •540 | •621 | •702 | 81 |
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| 546 | 7193 | 7272 | 7352 | 7431 | 7511 | 7590 | 7670 | 7749 | 7829 | 7908 | 79 |
| 547 | 7987 | 8067 | 8146 | 8225 | 8305 | 8384 | 8463 | 8543 | 8622 | 8701 | 79 |
| 548 | 8781 | 8860 | 8939 | 9018 | 9097 | 9177 | 9256 | 9335 | 9414 | 9493 | 79 |
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| 551 | 1152 | 1230 | 1309 | 1388 | 1467 | 1546 | 1624 | 1703 | 1782 | 1860 | 79 |
| 552 | 1939 | 2018 | 2096 | 2175 | 2254 | 2332 | 2411 | 2489 | 2568 | 2647 | 79 |
| 553 | 2725 | 2804 | 2882 | 2961 | 3039 | 3118 | 3196 | 3275 | 3353 | 3431 | 78 |
| 554 | 3510 | 3588 | 3667 | 3745 | 3823 | 3902 | 3980 | 4058 | 4136 | 4215 | 78 |
| 555 | 4293 | 4371 | 4449 | 4528 | 4606 | 4684 | 4762 | 4840 | 4919 | 4997 | 78 |
| 556 | 5075 | 5153 | 5231 | 5309 | 5387 | 5465 | 5543 | 5621 | 5699 | 5777 | 78 |
| 557 | 5855 | 5933 | 6011 | 6089 | 6167 | 6245 | 6323 | 6401 | 6479 | 6556 | 78 |
| 558 | 6634 | 6712 | 6790 | 6868 | 6945 | 7023 | 7101 | 7179 | 7256 | 7334 | 78 |
| 559 | 7412 | 7489 | 7567 | 7645 | 7722 | 7800 | 7878 | 7955 | 8033 | 8110 | 78 |
| 560 | 748188 | 8266 | 8343 | 8421 | 8498 | 8576 | 8653 | 8731 | 8808 | 8885 | 77 |
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| 562 | 9736 | 9814 | 9891 | 9968 | ••45 | •123 | •200 | •277 | •354 | •431 | 77 |
| 563 | 750508 | 0586 | 0663 | 0740 | 0817 | 0894 | 0971 | 1048 | 1125 | 1202 | 77 |
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| 565 | 2048 | 2125 | 2202 | 2279 | 2356 | 2433 | 2509 | 2586 | 2663 | 2740 | 77 |
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| 567 | 3583 | 3660 | 3736 | 3813 | 3889 | 3966 | 4042 | 4119 | 4195 | 4272 | 77 |
| 568 | 4348 | 4425 | 4501 | 4578 | 4654 | 4730 | 4807 | 4883 | 4960 | 5036 | 76 |
| 569 | 5112 | 5189 | 5265 | 5341 | 5417 | 5494 | 5570 | 5646 | 5722 | 5799 | 76 |
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| 571 | 6636 | 6712 | 6788 | 6864 | 6940 | 7016 | 7092 | 7168 | 7244 | 7320 | 76 |
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| 573 | 8155 | 8230 | 8306 | 8382 | 8458 | 8533 | 8609 | 8685 | 8761 | 8836 | 76 |
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| 578 | 1928 | 2003 | 2078 | 2153 | 2228 | 2303 | 2378 | 2453 | 2529 | 2604 | 75 |
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| 583 | 5669 | 5743 | 5818 | 5892 | 5966 | 6041 | 6115 | 6190 | 6264 | 6338 | 74 |
| 584 | 6413 | 6487 | 6562 | 6636 | 6710 | 6785 | 6859 | 6933 | 7007 | 7082 | 74 |
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| 587 | 8638 | 8712 | 8786 | 8860 | 8934 | 9008 | 9082 | 9156 | 9230 | 9303 | 74 |
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| 590 | 770852 | 0926 | 0999 | 1073 | 1146 | 1220 | 1293 | 1367 | 1440 | 1514 | 74 |
| 591 | 1587 | 1661 | 1734 | 1808 | 1881 | 1955 | 2028 | 2102 | 2175 | 2248 | 73 |
| 592 | 2322 | 2395 | 2468 | 2542 | 2615 | 2688 | 2762 | 2835 | 2908 | 2981 | 73 |
| 593 | 3055 | 3128 | 3201 | 3274 | 3348 | 3421 | 3494 | 3567 | 3640 | 3713 | 73 |
| 594 | 3786 | 3860 | 3933 | 4006 | 4079 | 4152 | 4225 | 4298 | 4371 | 4444 | 73 |
| 595 | 4517 | 4590 | 4663 | 4736 | 4809 | 4882 | 4955 | 5028 | 5100 | 5173 | 73 |
| 596 | 5246 | 5319 | 5392 | 5465 | 5538 | 5610 | 5683 | 5756 | 5829 | 5902 | 73 |
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| 599 | 7427 | 7499 | 7572 | 7644 | 7717 | 7789 | 7862 | 7934 | 8006 | 8079 | 72 |
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| 603 | 780317 | 0389 | 0461 | 0533 | 0605 | 0677 | 0749 | 0821 | 0893 | 0965 | 72 |
| 604 | 1037 | 1109 | 1181 | 1253 | 1324 | 1396 | 1468 | 1540 | 1612 | 1684 | 72 |
| 605 | 1755 | 1827 | 1899 | 1971 | 2042 | 2114 | 2186 | 2258 | 2329 | 2401 | 72 |
| 606 | 2473 | 2544 | 2616 | 2688 | 2759 | 2831 | 2902 | 2974 | 3046 | 3117 | 72 |
| 607 | 3189 | 3260 | 3332 | 3403 | 3475 | 3546 | 3618 | 3689 | 3761 | 3832 | 71 |
| 608 | 3904 | 3975 | 4046 | 4118 | 4189 | 4261 | 4332 | 4403 | 4475 | 4546 | 71 |
| 609 | 4617 | 4689 | 4760 | 4831 | 4902 | 4974 | 5045 | 5116 | 5187 | 5259 | 71 |
| 610 | 785330 | 5401 | 5472 | 5543 | 5615 | 5686 | 5757 | 5828 | 5899 | 5970 | 71 |
| 611 | 6041 | 6112 | 6183 | 6254 | 6325 | 6396 | 6467 | 6538 | 6609 | 6680 | 71 |
| 612 | 6751 | 6822 | 6893 | 6964 | 7035 | 7106 | 7177 | 7248 | 7319 | 7390 | 71 |
| 613 | 7460 | 7531 | 7602 | 7673 | 7744 | 7815 | 7885 | 7956 | 8027 | 8098 | 71 |
| 614 | 8168 | 8239 | 8310 | 8381 | 8451 | 8522 | 8593 | 8663 | 8734 | 8804 | 71 |
| 615 | 8875 | 8946 | 9016 | 9087 | 9157 | 9228 | 9299 | 9369 | 9440 | 9510 | 71 |
| 616 | 9581 | 9651 | 9722 | 9792 | 9863 | 9933 | •••4 | ••74 | •144 | •215 | 70 |
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| 624 | 5185 | 5254 | 5324 | 5393 | 5463 | 5532 | 5602 | 5672 | 5741 | 5811 | 70 |
| 625 | 5880 | 5949 | 6019 | 6088 | 6158 | 6227 | 6297 | 6366 | 6436 | 6505 | 69 |
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| 629 | 8651 | 8720 | 8789 | 8858 | 8927 | 8996 | 9065 | 9134 | 9203 | 9272 | 69 |
| 630 | 799341 | 9409 | 9478 | 9547 | 9616 | 9685 | 9754 | 9823 | 9892 | 9961 | 69 |
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| 633 | 1404 | 1472 | 1541 | 1609 | 1678 | 1747 | 1815 | 1884 | 1952 | 2021 | 69 |
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| 653 | 4913 | 4980 | 5046 | 5113 | 5179 | 5246 | 5312 | 5378 | 5445 | 5511 | 66 |
| 654 | 5578 | 5644 | 5711 | 5777 | 5843 | 5910 | 5976 | 6042 | 6109 | 6175 | 66 |
| 655 | 6241 | 6308 | 6374 | 6440 | 6506 | 6573 | 6639 | 6705 | 6771 | 6838 | 66 |
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| 657 | 7565 | 7631 | 7698 | 7764 | 7830 | 7896 | 7962 | 8028 | 8094 | 8160 | 66 |
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| 659 | 8885 | 8951 | 9017 | 9083 | 9149 | 9215 | 9281 | 9346 | 9412 | 9478 | 66 |
| 660 | 819544 | 9610 | 9676 | 9741 | 9807 | 9873 | 9939 | ••04 | ••70 | •136 | 66 |
| 661 | 820201 | 0267 | 0333 | 0399 | 0464 | 0530 | 0595 | 0661 | 0727 | 0792 | 66 |
| 662 | 0858 | 0924 | 0989 | 1055 | 1120 | 1186 | 1251 | 1317 | 1382 | 1448 | 66 |
| 663 | 1514 | 1579 | 1645 | 1710 | 1775 | 1841 | 1906 | 1972 | 2037 | 2103 | 65 |
| 664 | 2168 | 2233 | 2299 | 2364 | 2430 | 2495 | 2560 | 2626 | 2691 | 2756 | 65 |
| 665 | 2822 | 2887 | 2952 | 3018 | 3083 | 3148 | 3213 | 3279 | 3344 | 3409 | 65 |
| 666 | 3474 | 3539 | 3605 | 3670 | 3735 | 3800 | 3865 | 3930 | 3996 | 4061 | 65 |
| 667 | 4126 | 4191 | 4256 | 4321 | 4386 | 4451 | 4516 | 4581 | 4646 | 4711 | 65 |
| 668 | 4776 | 4841 | 4906 | 4971 | 5036 | 5101 | 5166 | 5231 | 5296 | 5361 | 65 |
| 669 | 5426 | 5491 | 5556 | 5621 | 5686 | 5751 | 5815 | 5880 | 5945 | 6010 | 65 |
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| 672 | 7369 | 7434 | 7499 | 7563 | 7628 | 7692 | 7757 | 7821 | 7886 | 7951 | 65 |
| 673 | 8015 | 8080 | 8144 | 8209 | 8273 | 8338 | 8402 | 8467 | 8531 | 8595 | 64 |
| 674 | 8660 | 8724 | 8789 | 8853 | 8918 | 8982 | 9046 | 9111 | 9175 | 9239 | 64 |
| 675 | 9304 | 9368 | 9432 | 9497 | 9561 | 9625 | 9690 | 9754 | 9818 | 9882 | 64 |
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| 679 | 1870 | 1934 | 1998 | 2062 | 2126 | 2189 | 2253 | 2317 | 2381 | 2445 | 64 |
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| 681 | 3147 | 3211 | 3275 | 3338 | 3402 | 3466 | 3530 | 3593 | 3657 | 3721 | 64 |
| 682 | 3784 | 3848 | 3912 | 3975 | 4039 | 4103 | 4166 | 4230 | 4294 | 4357 | 64 |
| 683 | 4421 | 4484 | 4548 | 4611 | 4675 | 4739 | 4802 | 4866 | 4929 | 4993 | 64 |
| 684 | 5056 | 5120 | 5183 | 5247 | 5310 | 5373 | 5437 | 5500 | 5564 | 5627 | 63 |
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| 694 | 1359 | 1422 | 1485 | 1547 | 1610 | 1672 | 1735 | 1797 | 1860 | 1922 | 63 |
| 695 | 1985 | 2047 | 2110 | 2172 | 2235 | 2297 | 2360 | 2422 | 2484 | 2547 | 62 |
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| 705 | 8189 | 8251 | 8312 | 8374 | 8435 | 8497 | 8559 | 8620 | 8682 | 8743 | 62 |
| 706 | 8805 | 8866 | 8928 | 8989 | 9051 | 9112 | 9174 | 9235 | 9297 | 9358 | 61 |
| 707 | 9419 | 9481 | 9542 | 9604 | 9665 | 9726 | 9788 | 9849 | 9911 | 9972 | 61 |
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| 722 | 8537 | 8597 | 8657 | 8718 | 8778 | 8838 | 8898 | 8958 | 9018 | 9078 | 60 |
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| 730 | 863323 | 3382 | 3442 | 3501 | 3561 | 3620 | 3680 | 3739 | 3799 | 3858 | 59 |
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| 766 | 4229 | 4285 | 4342 | 4399 | 4455 | 4512 | 4569 | 4625 | 4682 | 4739 | 57 |
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| 774 | 8741 | 8797 | 8853 | 8909 | 8965 | 9021 | 9077 | 9134 | 9190 | 9246 | 56 |
| 775 | 9302 | 9358 | 9414 | 9470 | 9526 | 9582 | 9638 | 9694 | 9750 | 9806 | 56 |
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| 779 | 1537 | 1593 | 1649 | 1705 | 1760 | 1816 | 1872 | 1928 | 1983 | 2039 | 56 |
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| 798 | 2003 | 2057 | 2112 | 2166 | 2221 | 2275 | 2329 | 2384 | 2438 | 2492 | 54 |
| 799 | 2547 | 2601 | 2655 | 2710 | 2764 | 2818 | 2873 | 2927 | 2981 | 3036 | 54 |
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| 801 | 3633 | 3687 | 3741 | 3795 | 3849 | 3904 | 3958 | 4012 | 4066 | 4120 | 54 |
| 802 | 4174 | 4229 | 4283 | 4337 | 4391 | 4445 | 4499 | 4553 | 4607 | 4661 | 54 |
| 803 | 4716 | 4770 | 4824 | 4878 | 4932 | 4986 | 5040 | 5094 | 5148 | 5202 | 54 |
| 804 | 5256 | 5310 | 5364 | 5418 | 5472 | 5526 | 5580 | 5634 | 5688 | 5742 | 54 |
| 805 | 5796 | 5850 | 5904 | 5958 | 6012 | 6066 | 6119 | 6173 | 6227 | 6281 | 54 |
| 806 | 6335 | 6389 | 6443 | 6497 | 6551 | 6604 | 6658 | 6712 | 6766 | 6820 | 54 |
| 807 | 6874 | 6927 | 6981 | 7035 | 7089 | 7143 | 7196 | 7250 | 7304 | 7358 | 54 |
| 808 | 7411 | 7465 | 7519 | 7573 | 7626 | 7680 | 7734 | 7787 | 7841 | 7895 | 54 |
| 809 | 7949 | 8002 | 8056 | 8110 | 8163 | 8217 | 8270 | 8324 | 8378 | 8431 | 54 |
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| 814 | 0624 | 0678 | 0731 | 0784 | 0838 | 0891 | 0944 | 0998 | 1051 | 1104 | 53 |
| 815 | 1158 | 1211 | 1264 | 1317 | 1371 | 1424 | 1477 | 1530 | 1584 | 1637 | 53 |
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| 822 | 4872 | 4925 | 4977 | 5030 | 5083 | 5136 | 5189 | 5241 | 5294 | 5347 | 53 |
| 823 | 5400 | 5453 | 5505 | 5558 | 5611 | 5664 | 5716 | 5769 | 5822 | 5875 | 53 |
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| 898 | 3276 | 3325 | 3373 | 3421 | 3470 | 3518 | 3566 | 3615 | 3663 | 3711 | 48 |
| 899 | 3760 | 3808 | 3856 | 3905 | 3953 | 4001 | 4049 | 4098 | 4146 | 4194 | 48 |
| 900 | 954243 | 4291 | 4339 | 4387 | 4435 | 4484 | 4532 | 4580 | 4628 | 4677 | 48 |
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| 902 | 5207 | 5255 | 5303 | 5351 | 5399 | 5447 | 5495 | 5543 | 5592 | 5640 | 48 |
| 903 | 5688 | 5736 | 5784 | 5832 | 5880 | 5928 | 5976 | 6024 | 6072 | 6120 | 48 |
| 904 | 6168 | 6216 | 6265 | 6313 | 6361 | 6409 | 6457 | 6505 | 6553 | 6601 | 48 |
| 905 | 6649 | 6697 | 6745 | 6793 | 6840 | 6888 | 6936 | 6984 | 7032 | 7080 | 48 |
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| 909 | 8564 | 8612 | 8659 | 8707 | 8755 | 8803 | 8850 | 8898 | 8946 | 8994 | 48 |
| 910 | 959041 | 9089 | 9137 | 9185 | 9232 | 9280 | 9328 | 9375 | 9423 | 9471 | 48 |
| 911 | 9518 | 9566 | 9614 | 9661 | 9709 | 9757 | 9804 | 9852 | 9900 | 9947 | 48 |
| 912 | 9995 | ••42 | ••90 | •138 | •185 | •233 | •280 | •328 | •376 | •423 | 48 |
| 913 | 960471 | 0518 | 0566 | 0613 | 0661 | 0709 | 0756 | 0804 | 0851 | 0899 | 48 |
| 914 | 0946 | 0994 | 1041 | 1089 | 1136 | 1184 | 1231 | 1279 | 1326 | 1374 | 47 |
| 915 | 1421 | 1469 | 1516 | 1563 | 1611 | 1658 | 1706 | 1753 | 1801 | 1848 | 47 |
| 916 | 1895 | 1943 | 1990 | 2038 | 2085 | 2132 | 2180 | 2227 | 2275 | 2322 | 47 |
| 917 | 2369 | 2417 | 2464 | 2511 | 2559 | 2606 | 2653 | 2701 | 2748 | 2795 | 47 |
| 918 | 2843 | 2890 | 2937 | 2985 | 3032 | 3079 | 3126 | 3174 | 3221 | 3268 | 47 |
| 919 | 3316 | 3363 | 3410 | 3457 | 3504 | 3552 | 3599 | 3646 | 3693 | 3741 | 47 |
| 920 | 963788 | 3835 | 3882 | 3929 | 3977 | 4024 | 4071 | 4118 | 4165 | 4212 | 47 |
| 921 | 4260 | 4307 | 4354 | 4401 | 4448 | 4495 | 4542 | 4590 | 4637 | 4684 | 47 |
| 922 | 4731 | 4778 | 4825 | 4872 | 4919 | 4966 | 5013 | 5061 | 5108 | 5155 | 47 |
| 923 | 5202 | 5249 | 5296 | 5343 | 5390 | 5437 | 5484 | 5531 | 5578 | 5625 | 47 |
| 924 | 5672 | 5719 | 5766 | 5813 | 5860 | 5907 | 5954 | 6001 | 6048 | 6095 | 47 |
| 925 | 6142 | 6189 | 6236 | 6283 | 6329 | 6376 | 6423 | 6470 | 6517 | 6564 | 47 |
| 926 | 6611 | 6658 | 6705 | 6752 | 6799 | 6845 | 6892 | 6939 | 6986 | 7033 | 47 |
| 927 | 7080 | 7127 | 7173 | 7220 | 7267 | 7314 | 7361 | 7408 | 7454 | 7501 | 47 |
| 928 | 7548 | 7595 | 7642 | 7688 | 7735 | 7782 | 7829 | 7875 | 7922 | 7969 | 47 |
| 929 | 8016 | 8062 | 8109 | 8156 | 8203 | 8249 | 8296 | 8343 | 8390 | 8436 | 47 |
| 930 | 968483 | 8530 | 8576 | 8623 | 8670 | 8716 | 8763 | 8810 | 8856 | 8903 | 47 |
| 931 | 8950 | 8996 | 9043 | 9090 | 9136 | 9183 | 9229 | 9276 | 9323 | 9369 | 47 |
| 932 | 9416 | 9463 | 9509 | 9556 | 9602 | 9649 | 9695 | 9742 | 9789 | 9835 | 47 |
| 933 | 9882 | 9928 | 9975 | ••21 | ••68 | •114 | •161 | •207 | •254 | •300 | 47 |
| 934 | 970347 | 0393 | 0440 | 0486 | 0533 | 0579 | 0626 | 0672 | 0719 | 0765 | 46 |
| 935 | 0812 | 0858 | 0904 | 0951 | 0997 | 1044 | 1090 | 1137 | 1183 | 1229 | 46 |
| 936 | 1276 | 1322 | 1369 | 1415 | 1461 | 1508 | 1554 | 1601 | 1647 | 1693 | 46 |
| 937 | 1740 | 1786 | 1832 | 1879 | 1925 | 1971 | 2018 | 2064 | 2110 | 2157 | 46 |
| 938 | 2203 | 2249 | 2295 | 2342 | 2388 | 2434 | 2481 | 2527 | 2573 | 2619 | 46 |
| 939 | 2666 | 2712 | 2758 | 2804 | 2851 | 2897 | 2943 | 2989 | 3035 | 3082 | 46 |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 940 | 973128 | 3174 | 3220 | 3266 | 3313 | 3359 | 3405 | 3451 | 3497 | 3543 | 46 |
| 941 | 3590 | 3636 | 3682 | 3728 | 3774 | 3820 | 3866 | 3913 | 3959 | 4005 | 46 |
| 942 | 4051 | 4097 | 4143 | 4189 | 4235 | 4281 | 4327 | 4374 | 4420 | 4466 | 46 |
| 943 | 4512 | 4558 | 4604 | 4650 | 4696 | 4742 | 4788 | 4834 | 4880 | 4926 | 46 |
| 944 | 4972 | 5018 | 5064 | 5110 | 5156 | 5202 | 5248 | 5294 | 5340 | 5386 | 46 |
| 945 | 5432 | 5478 | 5524 | 5570 | 5616 | 5662 | 5707 | 5753 | 5799 | 5845 | 46 |
| 946 | 5891 | 5937 | 5983 | 6029 | 6075 | 6121 | 6167 | 6212 | 6258 | 6304 | 46 |
| 947 | 6350 | 6396 | 6442 | 6488 | 6533 | 6579 | 6625 | 6671 | 6717 | 6763 | 46 |
| 948 | 6808 | 6854 | 6900 | 6946 | 6992 | 7037 | 7083 | 7129 | 7175 | 7220 | 46 |
| 949 | 7266 | 7312 | 7358 | 7403 | 7449 | 7495 | 7541 | 7586 | 7632 | 7678 | 46 |
| 950 | 977724 | 7769 | 7815 | 7861 | 7906 | 7952 | 7998 | 8043 | 8089 | 8135 | 46 |
| 951 | 8181 | 8226 | 8272 | 8317 | 8363 | 8409 | 8454 | 8500 | 8546 | 8591 | 46 |
| 952 | 8637 | 8683 | 8728 | 8774 | 8819 | 8865 | 8911 | 8956 | 9002 | 9047 | 46 |
| 953 | 9093 | 9138 | 9184 | 9230 | 9275 | 9321 | 9366 | 9412 | 9457 | 9503 | 46 |
| 954 | 9548 | 9594 | 9639 | 9685 | 9730 | 9776 | 9821 | 9867 | 9912 | 9958 | 46 |
| 955 | 980003 | 0049 | 0094 | 0140 | 0185 | 0231 | 0276 | 0322 | 0367 | 0412 | 45 |
| 956 | 0458 | 0503 | 0549 | 0594 | 0640 | 0685 | 0730 | 0776 | 0821 | 0867 | 45 |
| 957 | 0912 | 0957 | 1003 | 1048 | 1093 | 1139 | 1184 | 1229 | 1275 | 1320 | 45 |
| 958 | 1366 | 1411 | 1456 | 1501 | 1547 | 1592 | 1637 | 1683 | 1728 | 1773 | 45 |
| 959 | 1819 | 1864 | 1909 | 1954 | 2000 | 2045 | 2090 | 2135 | 2181 | 2226 | 45 |
| 960 | 982271 | 2316 | 2362 | 2407 | 2452 | 2497 | 2543 | 2588 | 2633 | 2678 | 45 |
| 961 | 2723 | 2769 | 2814 | 2859 | 2904 | 2949 | 2994 | 3040 | 3085 | 3130 | 45 |
| 962 | 3175 | 3220 | 3265 | 3310 | 3356 | 3401 | 3446 | 3491 | 3536 | 3581 | 45 |
| 963 | 3626 | 3671 | 3716 | 3762 | 3807 | 3852 | 3897 | 3942 | 3987 | 4032 | 45 |
| 964 | 4077 | 4122 | 4167 | 4212 | 4257 | 4302 | 4347 | 4392 | 4437 | 4482 | 45 |
| 965 | 4527 | 4572 | 4617 | 4662 | 4707 | 4752 | 4797 | 4842 | 4887 | 4932 | 45 |
| 966 | 4977 | 5022 | 5067 | 5112 | 5157 | 5202 | 5247 | 5292 | 5337 | 5382 | 45 |
| 967 | 5426 | 5471 | 5516 | 5561 | 5606 | 5651 | 5696 | 5741 | 5786 | 5830 | 45 |
| 968 | 5875 | 5920 | 5965 | 6010 | 6055 | 6100 | 6144 | 6189 | 6234 | 6279 | 45 |
| 969 | 6324 | 6369 | 6413 | 6458 | 6503 | 6548 | 6593 | 6637 | 6682 | 6727 | 45 |
| 970 | 986772 | 6817 | 6861 | 6906 | 6951 | 6996 | 7040 | 7085 | 7130 | 7175 | 45 |
| 971 | 7219 | 7264 | 7309 | 7353 | 7398 | 7443 | 7488 | 7532 | 7577 | 7622 | 45 |
| 972 | 7666 | 7711 | 7756 | 7800 | 7845 | 7890 | 7934 | 7979 | 8024 | 8068 | 45 |
| 973 | 8113 | 8157 | 8202 | 8247 | 8291 | 8336 | 8381 | 8425 | 8470 | 8514 | 45 |
| 974 | 8559 | 8604 | 8648 | 8693 | 8737 | 8782 | 8826 | 8871 | 8916 | 8960 | 45 |
| 975 | 9005 | 9049 | 9094 | 9138 | 9183 | 9227 | 9272 | 9316 | 9361 | 9405 | 45 |
| 976 | 9450 | 9494 | 9539 | 9583 | 9628 | 9672 | 9717 | 9761 | 9806 | 9850 | 44 |
| 977 | 9895 | 9939 | 9983 | ••28 | ••72 | •117 | •161 | •206 | •250 | •294 | 44 |
| 978 | 990339 | 0383 | 0428 | 0472 | 0516 | 0561 | 0605 | 0650 | 0694 | 0738 | 44 |
| 979 | 0783 | 0827 | 0871 | 0916 | 0960 | 1004 | 1049 | 1093 | 1137 | 1182 | 44 |
| 980 | 991226 | 1270 | 1315 | 1359 | 1403 | 1448 | 1492 | 1536 | 1580 | 1625 | 44 |
| 981 | 1669 | 1713 | 1758 | 1802 | 1846 | 1890 | 1935 | 1979 | 2023 | 2067 | 44 |
| 982 | 2111 | 2156 | 2200 | 2244 | 2288 | 2333 | 2377 | 2421 | 2465 | 2509 | 44 |
| 983 | 2554 | 2598 | 2642 | 2686 | 2730 | 2774 | 2819 | 2863 | 2907 | 2951 | 44 |
| 984 | 2995 | 3039 | 3083 | 3127 | 3172 | 3216 | 3260 | 3304 | 3348 | 3392 | 44 |
| 985 | 3436 | 3480 | 3524 | 3568 | 3613 | 3657 | 3701 | 3745 | 3789 | 3833 | 44 |
| 986 | 3877 | 3921 | 3965 | 4009 | 4053 | 4097 | 4141 | 4185 | 4229 | 4273 | 44 |
| 987 | 4317 | 4361 | 4405 | 4449 | 4493 | 4537 | 4581 | 4625 | 4669 | 4713 | 44 |
| 988 | 4757 | 4801 | 4845 | 4889 | 4933 | 4977 | 5021 | 5065 | 5108 | 5152 | 44 |
| 989 | 5196 | 5240 | 5284 | 5328 | 5372 | 5416 | 5460 | 5504 | 5547 | 5591 | 44 |
| 990 | 995635 | 5679 | 5723 | 5767 | 5811 | 5854 | 5898 | 5942 | 5986 | 6030 | 44 |
| 991 | 6074 | 6117 | 6161 | 6205 | 6249 | 6293 | 6337 | 6380 | 6424 | 6468 | 44 |
| 992 | 6512 | 6555 | 6599 | 6643 | 6687 | 6731 | 6774 | 6818 | 6862 | 6906 | 44 |
| 993 | 6949 | 6993 | 7037 | 7080 | 7124 | 7168 | 7212 | 7255 | 7299 | 7343 | 44 |
| 994 | 7386 | 7430 | 7474 | 7517 | 7561 | 7605 | 7648 | 7692 | 7736 | 7779 | 44 |
| 995 | 7823 | 7867 | 7910 | 7954 | 7998 | 8041 | 8085 | 8129 | 8172 | 8216 | 44 |
| 996 | 8259 | 8303 | 8347 | 8390 | 8434 | 8477 | 8521 | 8564 | 8608 | 8652 | 44 |
| 997 | 8695 | 8739 | 8782 | 8826 | 8869 | 8913 | 8956 | 9000 | 9043 | 9087 | 44 |
| 998 | 9131 | 9174 | 9218 | 9261 | 9305 | 9348 | 9392 | 9435 | 9479 | 9522 | 44 |
| 999 | 9565 | 9609 | 9652 | 9696 | 9739 | 9783 | 9826 | 9870 | 9913 | 9957 | 43 |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |

Bennie
Henkle